MATHEMATICAL TOOLS

0.1 ALGEBRA

Common Identities

(i) $(a+b)^2 = a^2 + 2ab + b^2 = (a-b)^2 + 4ab$

(ii) $(a-b)^2 = a^2 - 2ab + b^2 = (a+b)^2 - 4ab$

(iii) $a^2 - b^2 = (a+b)(a-b)$

(iv) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$= a^3 + b^3 + 3ab(a+b)$

(v) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
$= a^3 - b^3 - 3ab(a-b)$

(vi) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
$= (a+b)^3 - 3ab(a+b)$

(vii) $a^3 - b^3 = (a-b)(a^2 + ab + b^3)$
$= (a-b)^3 + 3ab(a-b)$

(viii) $(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$

(ix) $(a+b)^2 - (a-b)^2 = 4ab$

(x) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

Quadratic Equation

An equation of second degree is called a quadratic equation. It is of the form :

$$ax^2 + bx + c = 0$$

The roots of a quadratic equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 1. *Solve the equation :* $6x^2 - 13x + 6 = 0.$

Solution. Here $a = 6, \quad b = -13, \quad c = 6$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+13 \pm \sqrt{169 - 4 \times 6 \times 6}}{2 \times 6}$$

$$= \frac{13 \pm \sqrt{169 - 144}}{12} = \frac{13 \pm 5}{12} = \frac{18}{12}, \frac{8}{12}$$

or $\quad x = \frac{3}{2}, \frac{2}{3}.$

Binomial Theorem

If n is any integer, positive or negative, or a fraction and x is any real number, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

where $2! = 2 \times 1, \quad 3! = 3 \times 2 \times 1$

In general, $\quad n! = n(n-1)(n-2)\ldots 3 \times 2 \times 1.$

If $|x| << 1$, then $(1+x)^n \simeq 1 + nx.$

EXAMPLE 2. *The acceleration due to gravity at a height h above the surface of the earth (radius = R) is given by*

$$g' = \frac{gR^2}{(R+h)^2}$$

If $h << R$, *then show that :* $g' = g\left(1 - \frac{2h}{R}\right).$

Solution. $g' = \dfrac{gR^2}{(R+h)^2}$

$$= \frac{gR^2}{R^2\left(1+\frac{h}{R}\right)^2} = g\left(1+\frac{h}{R}\right)^{-2}$$

Using binomial theorem,

$$g' = g\left[1+(-2)\frac{h}{R} + \frac{(-2)(-3)}{2!}\left(\frac{h}{R}\right)^2 + ...\right]$$

As $h << R$, h/R is very small, so terms containing higher powers of h/R can be neglected.

$$\therefore \qquad g' = g\left(1-\frac{2h}{R}\right).$$

0.2 ▼ MENSURATION

Important Formulae

Circumference of a circle $= 2\pi r = \pi D$

Area of a circle $= \pi r^2 = \dfrac{\pi D^2}{4}$

Surface area of a sphere $= 4\pi r^2 = \pi D^2$

Volume of a sphere $= \dfrac{4}{3}\pi r^3$

Surface area of a cylinder

$= 2\pi r^2 + 2\pi rl = 2\pi r(r+l)$

Volume of a cylinder $= \pi r^2 l$

Curved surface area of a cone $= \pi rl$

Volume of a cone $= \dfrac{1}{3}\pi r^2 h$

Volume of a cube $= (\text{side})^3$

Surface area of a cube $= 6 \times (\text{side})^2$.

0.3 ▼ TRIGONOMETRY

Systems of Measurement of an Angle

(i) *Sexagesimal system.* In this system,

1 right angle $= 90°$ (degree)

$1° = 60'$ (minute)

$1' = 60''$ (second)

(ii) *Centesimal system.* In this system

1 right angle $= 100^g$ (grade)

$1^g = 100'$ (minute)

$1' = 100''$ (second)

(iii) *Circular system.* In this system, the unit of angle is radian.

One radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.

If l is the length of an arc and θ is the angle subtended at the centre of the circle as shown in Fig. 0.1, then

$$\theta = \frac{\text{Arc}}{\text{Radius}} = \frac{l}{r} \text{ radian}$$

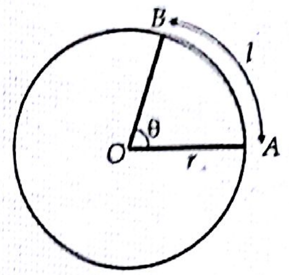

Fig. 0.1

Angle subtended at the centre of the circle is

$$\theta = \frac{\text{Circumference}}{\text{Radius}} = \frac{2\pi r}{r} = 2\pi \text{ radian}$$

π radian $= 180° = 200^g$

1 radian $= 57°916'22'' = 63^g\, 63'\, 64''$

Trigonometrical Ratios

In right angled ΔOMP, of Fig. 0.2, $\angle OMP = 90°$ and $\angle POM = \theta$.

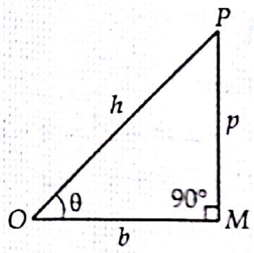

Fig. 0.2

We can define the trigonometric ratios as follows :

$$\text{sine } \theta = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{PM}{OP} = \sin\theta$$

$$\text{cosine } \theta = \frac{\text{base}}{\text{hypotenuse}} = \frac{OM}{OP} = \cos\theta$$

$$\text{tangent } \theta = \frac{\text{perpendicular}}{\text{base}} = \frac{PM}{OM} = \tan\theta$$

$$\text{cosecant } \theta = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{OP}{PM} = \text{cosec}\,\theta$$

$$\text{secant } \theta = \frac{\text{hypotenuse}}{\text{base}} = \frac{OP}{OM} = \sec\theta$$

$$\text{cotangent } \theta = \frac{\text{base}}{\text{perpendicular}} = \frac{OM}{PM} = \cot\theta$$

Fundamental Trigonometric Relations

1. $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$, $\quad \sec\theta = \dfrac{1}{\cos\theta}$, $\quad \cot\theta = \dfrac{1}{\tan\theta}$

2. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\quad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

3. $\sin^2\theta + \cos^2\theta = 1$, $\quad 1 + \tan^2\theta = \sec^2\theta$,

$$1 + \cot^2\theta = \text{cosec}^2\,\theta$$

T-ratios of Allied Angles

$\sin(-\theta) = -\sin\theta$ $\qquad$ $\text{cosec}(-\theta) = -\text{cosec}\,\theta$

$\cos(-\theta) = \cos\theta$ $\qquad$ $\sec(-\theta) = \sec\theta$

$\tan(-\theta) = -\tan\theta$ $\qquad$ $\cot\theta = -\cot\theta$

$\sin(90° - \theta) = \cos\theta$ $\qquad$ $\text{cosec}(90° - \theta) = \sec\theta$

$\cos(90° - \theta) = \sin\theta$ $\qquad$ $\sec(90° - \theta) = \text{cosec}\,\theta$

$\tan(90° - \theta) = \cot\theta$ $\qquad$ $\cot(90° - \theta) = \tan\theta$

$\sin(90° + \theta) = \cos\theta$ $\qquad$ $\text{cosec}(90° + \theta) = \sec\theta$

$\cos(90° + \theta) = -\sin\theta$ $\qquad$ $\sec(90° + \theta) = -\text{cosec}\,\theta$

$\tan(90° + \theta) = -\cot\theta$ $\qquad$ $\cot(90° + \theta) = -\tan\theta$

$\sin(180° - \theta) = \sin\theta$ $\qquad$ $\text{cosec}(180° - \theta) = \text{cosec}\,\theta$

$\cos(180° - \theta) = -\cos\theta$ $\qquad$ $\sec(180° - \theta) = -\sec\theta$

$\tan(180° - \theta) = -\tan\theta$ $\qquad$ $\cot(180° - \theta) = -\cot\theta$

$\sin(180° + \theta) = -\sin\theta$ $\qquad$ $\text{cosec}(180° + \theta) = -\text{cosec}\,\theta$

$\cos(180° + \theta) = -\cos\theta$ $\qquad$ $\sec(180° + \theta) = -\sec\theta$

$\tan(180° + \theta) = \tan\theta$ $\qquad$ $\cot(180° + \theta) = \cot\theta$

$\sin(270° - \theta) = -\cos\theta$ $\qquad$ $\text{cosec}(270° - \theta) = -\sec\theta$

$\cos(270° - \theta) = -\sin\theta$ $\qquad$ $\sec(270° - \theta) = -\text{cosec}\,\theta$

$\tan(270° - \theta) = \cot\theta$ $\qquad$ $\cot(270° - \theta) = \tan\theta$

$\sin(270° + \theta) = -\cos\theta$ $\qquad$ $\text{cosec}(270° + \theta) = -\sec\theta$

$\cos(270° + \theta) = \sin\theta$ $\qquad$ $\sec(270° + \theta) = \text{cosec}\,\theta$

$\tan(270° + \theta) = -\cot\theta$ $\qquad$ $\cot(270° + \theta) = -\tan\theta$

$\sin(360° - \theta) = -\sin\theta$ $\qquad$ $\text{cosec}(360° - \theta) = -\text{cosec}\,\theta$

$\cos(360° - \theta) = \cos\theta$ $\qquad$ $\sec(360° - \theta) = \sec\theta$

$\tan(360° - \theta) = -\tan\theta$ $\qquad$ $\cot(360° - \theta) = -\cot\theta$

Some Important Trigonometrical Formulae

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A - B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1 + \tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$$

$$= 2\cos^2 A - 1 = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$\sin(A + B) + \sin(A - B) = 2\sin A\cos B$$

$$\sin(A + B) - \sin(A - B) = 2\cos A\sin B$$

$$\cos(A + B) + \cos(A - B) = 2\cos A\cos B$$

$$\cos(A + B) - \cos(A - B) = -2\sin A\sin B$$

$$\sin C + \sin D = 2\sin\frac{C + D}{2}\cos\frac{C - D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C + D}{2}\sin\frac{C - D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C + D}{2}\cos\frac{C - D}{2}$$

$$\cos C - \cos D = -2\sin\frac{C + D}{2}\sin\frac{C - D}{2}$$

Values of Trigonometrical Ratios of Some Standard Angles

angle θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$-\infty$	0

0.4 ▼ LOGARITHMS

Definition of Logarithm

The logarithm of any number to a given base is the power to which base must be raised to obtain that number.

For example, $81 = 3^4$, we can say that the logarithm of 81 to the base 3 is equal to 4.

Symbolically, $\log_3 81 = 4$

In general, if $N = a^x$, then $\log_a N = x$.

The common logarithm of a number is the power to which 10 must be raised to obtain that number.

As $1000 = 10^3$ $\therefore$ $\log_{10} 1000 = 3$

As $a^\circ = 1$ $\therefore$ $\log_a 1 = 0$

As $a^1 = a$ $\therefore$ $\log_a a = 1$

Logarithmic Formulae

Product formula :

$$\log_a mn = \log_a m + \log_a n$$

Quotient formula :

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

Power formula :

$$\log_a m^n = n \log_a m$$

Base change formula :

$$\log_a m = \log_b m \times \log_a b$$

0.5 ▼ DIFFERENTIAL CALCULUS

Differential Coefficient

Let y be a function of x *i.e.*, $y = f(x)$

Suppose the value of x increases by a small amount Δx. Then the value of y also increases by a small amount, say Δy.

The ratio $\frac{\Delta y}{\Delta x}$ is called the *average rate of change of y with respect to x*.

When Δx approaches zero, the limiting value of $\frac{\Delta y}{\Delta x}$ is called *differential coefficient* or *derivative of y w.r.t. x* and is denoted by $\frac{dy}{dx}$.

Hence $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Physically, the *derivative* $\frac{dy}{dx}$ *gives the instantaneous rate of change of function y with respect to variable x.*

Some Important Results on Differentiation

(*i*) Let c be a constant. Then $\frac{d}{dx}(c) = 0$

(*ii*) $\frac{d}{dx}(cy) = c \cdot \frac{dy}{dx}$

(*iii*) $\frac{d}{dx}(x^n) = nx^{n-1}$

(*iv*) Let $y = u \pm v$, where u and v are functions of x.

Then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

(*v*) *Product Rule.*

Let $y = uv$, then

$$\frac{dy}{dx} = F.F. \frac{d}{dx}(S.F.) + S.F. \frac{d}{dx}(F.F.)$$

$$= u\frac{dv}{dx} + v\frac{du}{dx}$$

(*vi*) *Quotient rule.*

Let $y = \frac{u}{v}$, then

$$\frac{dy}{dx} = \frac{\text{Den}\frac{d}{dx}(\text{Num}) - (\text{Num})\frac{d}{dx}(\text{Den})}{(\text{Den})^2}$$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

(*vii*) *Chain rule.* Let y be a function of u and u be a function of x. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Let $y = u^n$. Then

$$\frac{d}{dx}(u^n) = nu^{n-1} \cdot \frac{du}{dx}$$

(*viii*) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(*ix*) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_e a$

(*x*) $\frac{d}{dx}(e^x) = e^x$

(*xi*) $\frac{d}{dx}(a^x) = a^x \log_e a$

(*xii*) $\frac{d}{dx}(\sin x) = \cos x$

(*xiii*) $\frac{d}{dx}(\cos x) = -\sin x$

(*xiv*) $\frac{d}{dx}(\tan x) = \sec^2 x$

(*xv*) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(*xvi*) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(*xvii*) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

EXAMPLE 3. *Find* $\frac{dy}{dx}$ *for the following functions :*

(i) $y = x^5 + x^3 + 10$ (ii) $y = x + \sqrt{x} + \frac{1}{\sqrt{x}}$

(iii) $y = 5x^4 + 3x^{3/2} + 6x$.

Solution.

(i) $y = x^5 + x^3 + 10$

$$\frac{dy}{dx} = \frac{d}{dx}(x^5) + \frac{d}{dx}(x^3) + \frac{d}{dx}(10)$$

$$= 5x^4 + 3x^2 + 0 = \mathbf{5x^4 + 3x^2}.$$

(ii) $y = x + \sqrt{x} + \frac{1}{\sqrt{x}} = x^1 + x^{1/2} + x^{-1/2}$

$$\frac{dy}{dx} = 1.x^0 + \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$$

$$= 1 + \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}.$$

(iii) $y = 5x^4 + 3x^{3/2} + 6x$

$$\frac{dy}{dx} = 5\frac{d}{dx}(x^4) + 3\frac{d}{dx}(x^{3/2}) + 6\frac{d}{dx}(x)$$

$$= 5 \times 4x^3 + 3 \times \frac{3}{2}x^{1/2} + 6 \times 1$$

$$= \mathbf{20x^3 + \frac{9}{2}\sqrt{x} + 6}.$$

EXAMPLE 4. *Differentiate the following functions :*

(i) $(3x^2 + 7)(6x + 3)$ (ii) $\frac{x^2 + 1}{x - 2}$

(iii) $\sqrt{4x^2 - 7}$.

Solution. (i) Let $y = (3x^2 + 7)(6x + 3)$

Using product rule, we get

$$\frac{dy}{dx} = (3x^2 + 7)\frac{d}{dx}(6x + 3) + (6x + 3)\frac{d}{dx}(3x^2 + 7)$$

$$= (3x^2 + 7)(6 + 0) + (6x + 3)(6x + 0)$$

$$= 18x^2 + 42 + 36x^2 + 18x = \mathbf{54x^2 + 18x + 42}.$$

(ii) $y = \frac{x^2 + 1}{x - 2}$

Using quotient rule, we get

$$\frac{dy}{dx} = \frac{(x-2)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x-2)}{(x-2)^2}$$

$$= \frac{(x-2)(2x+0) - (x^2+1)(1-0)}{(x-2)^2}$$

$$= \frac{2x^2 - 4x - x^2 - 1}{(x-2)^2} = \mathbf{\frac{x^2 - 4x - 1}{(x-2)^2}}.$$

(iii) $y = \sqrt{4x^2 - 7} = (4x^2 - 7)^{1/2}$

Using chain rule,

$$\frac{dy}{dx} = \frac{1}{2}(4x^2 - 7)^{-1/2}\frac{d}{dx}(4x^2 - 7)$$

$$= \frac{1}{2}(4x^2 - 7)^{-1/2}(8x - 0) = \mathbf{\frac{4x}{\sqrt{4x^2 - 7}}}.$$

EXAMPLE 5. *Find the differential coefficient of the following functions :*

(i) $\cos(ax^2 + b)$ (ii) $\tan^3 x$ (iii) $\frac{\sin x}{1 + \cos x}$

Solution. (i) Let $y = \cos(ax^2 + b)$

Then $\frac{dy}{dx} = -\sin(ax^2 + b)\frac{d}{dx}(ax^2 + b)$

$$= -\sin(ax^2 + b).2ax$$

$$= \mathbf{-2ax \sin(ax^2 + b)}.$$

(ii) Let $y = \tan^3 x = (\tan x)^3$

Then $\frac{dy}{dx} = 3(\tan x)^2\frac{d}{dx}(\tan x)$

$$= \mathbf{3\tan^2 x . \sec^2 x}.$$

(iii) Let $y = \frac{\sin x}{1 + \cos x}$

Then

$$\frac{dy}{dx} = \frac{(1+\cos x)\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)\cos x - \sin x(0 - \sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \mathbf{\frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}}.$$

EXAMPLE 6. *A particle is moving with a uniform acceleration. Its displacement at any instant t is given by* $s = 10t + 4.9t^2$. *What is (i) initial velocity (ii) velocity at t = 3 second and (iii) the uniform acceleration ?*

Solution. Displacement, $s = 10t + 4.9t^2$

Velocity, $v = \frac{ds}{dt} = 10 + 2 \times 4.9t$

(i) Initial velocity

(*i.e.*, velocity $t = 0$) $= 10 + 0 =$ **10 ms^{-1}**.

(ii) Velocity at $t = 3$ second is

$v = 10 + 2 \times 4.9 \times 3 =$ **39.4 ms^{-1}**.

(iii) Acceleration,

$a = \frac{dv}{dt} = \frac{d}{dt}(10 + 9.8t) = 0 + 9.8 =$ **9.8 ms^{-2}**

EXAMPLE 7. *A particle starts rotating from rest according to the formula,*

$$\theta = \frac{3t^3}{20} - \frac{t^2}{3}$$

where θ is in radian and t in second. Find the angular velocity ω and angular acceleration α at the end of 5 seconds.

Solution. Given $\theta = \frac{3t^3}{20} - \frac{t^2}{3}$

Angular velocity,

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{3t^3}{20} - \frac{t^2}{3}\right) = \frac{9t^2}{20} - \frac{2t}{3}$$

At $t = 5$ s,

$$\omega = \frac{9 \times 25}{20} - \frac{2 \times 5}{3} = \frac{475}{60} = \mathbf{7.92\ rad\ s^{-1}}.$$

Angular acceleration,

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{9t^2}{20} - \frac{2t}{3}\right) = \frac{18}{20}t - \frac{2}{3}$$

At $t = 5$ s,

$$\alpha = \frac{18 \times 5}{20} - \frac{2}{3} = \mathbf{3.83\ rad\ s^{-2}}.$$

EXAMPLE 8. *Show that power is the product of force and velocity.*

Solution. Work = Force × distance

or $W = Fs$

Power = Rate of doing work

$$= \frac{dW}{dt} = \frac{d}{dt}(Fs) = F\frac{ds}{dt} = \mathbf{F \cdot v}.$$

EXAMPLE 9. *A balloon is being filled by air so that its volume V is gradually increasing. Find the rate of increase of volume with radius r when r = 2 units.*

Solution. The volume of spherical balloon is

$$V = \frac{4}{3}\pi r^3$$

The rate of increase of volume V *w.r.t.* the radius r is

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)$$

or

$$\frac{dV}{dr} = \frac{4}{3}\pi \cdot \frac{d}{dr}(r^3) = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$$

When $r = 2$

$$\frac{dV}{dr} = 4\pi(2)^2 = \mathbf{16\pi}.$$

EXAMPLE 10. *For a particle executing simple harmonic motion, the displacement from the mean position is given by $y = a\sin(\omega t + \phi)$; where a, ω and φ are constants. Find the velocity and acceleration of the particle at any instant t.*

Solution. Displacement, $y = a\sin(\omega t + \phi)$

Velocity,

$$v = \frac{dy}{dt} = \frac{d}{dt}[a\sin(\omega t + \phi)]$$

$$= a\cos(\omega t + \phi)\frac{d}{dt}(\omega t + \phi)$$

$$= \omega a\cos(\omega t + \phi)$$

Acceleration,

$$a = \frac{dv}{dt} = \frac{d}{dt}[\omega a\cos(\omega t + \phi)]$$

$$= -\omega a\sin(\omega t + \phi)\frac{d}{dt}(\omega t + \phi)$$

$$= -\omega^2 a\sin(\omega t + \phi).$$

0.6 INTEGRAL CALCULUS

Integration

Integration is the reverse process of differentiation. It is the process of finding a function whose derivative is given. If derivative of function $f(x)$ *w.r.t.* x is $f'(x)$, then integration of $f'(x)$ *w.r.t.* x is $f(x)$. Symbolically, we can say

if $\frac{d}{dx}[f(x)] = f'(x)$, then $\int f'(x)dx = f(x)$.

Some Standard Elementary Integrals

Some standard elementary integrals alongwith their results on differentiation are as follows :

Differentiation	Integration
1. $\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{(n+1)} + c$, provided $n \neq -1$. Here c is constant of integration.
2. $\frac{d}{dx}(x) = 1$	$\int dx = x + c$
3. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$	$\int \frac{dx}{x} = \log_e x + c$
4. $\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x \cdot dx = \sin x + c$
5. $\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x \cdot dx = -\cos x + c$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x \cdot dx = \tan x + c$
7. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$	$\int \text{cosec}^2 x \cdot dx = -\cot x + c$
8. $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$	$\int \sec x \cdot \tan x\,dx = \sec x + c$

Differentiation	Integration
9. $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x . \cot x$	$\int \text{cosec } x . \cot x \, dx = -\text{cosec } x + c$
10. $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$	$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c.$
11. $\frac{d}{dx}\log_e(ax+b) = \frac{a}{(ax+b)}$	$\int \frac{dx}{(ax+b)} = \frac{1}{a}\log_e(ax+b) + c$
12. $\frac{d}{dx}(e^x) = e^x$	$\int e^x \cdot dx = e^x + c$
13. $\frac{d}{dx}(a^x) = a^x \cdot \log_e a$	$\int a^x \cdot dx = \frac{a^x}{\log_e a} = a^x . \log_a e + c$
14. $y = u \pm v \pm w$; $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx}$	$\int (u \pm v \pm w)\, dx = \int u \cdot dx \pm \int v dx \pm \int w dx + c$

Definite integral

When an integral is defined between two definite limits a and b, it is said to be a definite integral. It is given by

$$\int_a^b f(x)dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$$

where $\phi(x)$ is the integral of $f(x)$. Here a and b are the lower and upper limits of integration.

EXAMPLE 11. *Integrate :* $x^2 - \cos x + \frac{1}{x}$.

Solution. $\int \left(x^2 - \cos x + \frac{1}{x}\right) dx$

$$= \int x^2 \, dx - \int \cos x \, dx + \int \frac{1}{x} \, dx$$

$$= \frac{x^3}{3} - \sin x + \log x + c.$$

EXAMPLE 12. *Evaluate* $\int_0^{\pi/6} \sec^2 x dx$.

Solution. $\int_0^{\pi/6} \sec^2 x dx = [\tan x]_0^{\pi/6}$

$$= \tan\frac{\pi}{6} - \tan 0 = \frac{1}{\sqrt{3}} - 0 = \frac{1}{\sqrt{3}}.$$

EXAMPLE 13. *Find the value of* $\int_R^{\infty} \frac{GMmdx}{x^2}$ *; where G, M and m are constants.*

Solution. $\int_R^{\infty} \frac{GMm}{x^2} dx = GMm \int_R^{\infty} x^{-2} dx$

$$= GMm\left[\frac{x^{-1}}{-1}\right]_R^{\infty} = -GMm\left[\frac{1}{x}\right]_R^{\infty}$$

$$= -GMm\left[\frac{1}{\infty} - \frac{1}{R}\right]$$

$$= -GMm\left[0 - \frac{1}{R}\right] = \frac{GMm}{R}.$$

EXAMPLE 14. *Find the value of* $\int_0^x F \, dx$ *; where* $F = kx$.

Solution. $\int_0^x F dx = \int_0^x kx dx$

$$= k\int_0^x x dx = k\left[\frac{x^2}{2}\right]_0^x$$

$$= k\left[\frac{x^2}{2} - 0\right] = \frac{1}{2}kx^2.$$

EXAMPLE 15. *Find the value of* $\int_{V_1}^{V_2} \frac{1}{V} dV$.

Solution. $\int_{V_1}^{V_2} \frac{1}{V} dV = [\log_e V]_{V_1}^{V_2}$

$$= [\log_e V_2 - \log_e V_1]$$

$$= \log_e \frac{V_2}{V_1}.$$

EXAMPLE 16. *Evaluate* $\int_{-l/2}^{+l/2} \frac{M}{l} . x^2 dx$*; where M and l are constants.*

Solution.

$$\int_{-l/2}^{+l/2} \frac{M}{l} . x^2 \, dx = \frac{M}{l}\int_{-l/2}^{+l/2} x^2 dx$$

$$= \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{+l/2}$$

$$= \frac{M}{3l}\left[\left(\frac{l}{2}\right)^3 - \left(-\frac{l}{2}\right)^3\right]$$

$$= \frac{M}{3l}\cdot\left[\frac{l^3}{8} + \frac{l^3}{8}\right] = \frac{M}{3l}\cdot\frac{2l^3}{8} = \frac{Ml^2}{12}.$$