MOTION OF HEAVENLY BODIES

Geocentric model

The celestial bodies have been an object of interest to scientists all through the ages. The planets and stars appear stuck to the inside of the huge hemispherical surface called celestial sphere. The first astronomical observation was perhaps made by the Chinese as far back as 2000 B.C., though the Babylonian astronomers are credited with having mapped out constallations even earlier. The first authoritative treatise on the subject was due to *Ptolomy* in about 100 A.D. According to him earth was considered as the centre of the universe. The stars including the sun revolved round the earth. The planets revolved in small circle called *epicycle* and the centre of these epicycles moved in larger circles called *deferents* around the earth. A similar geocentric theory was advanced by Indian astronomers in the 5th century.

Heliocentric model

In 1543 Nicholaus Copernicus stated that all planets moved round the Sun in circles and the Sun was fixed. The Italian scientist Galileo subscribed to this model on the basis of experimental studies using his telescope. Tycho Brahe (1546–1601) made a series of observations regarding the motion of planets and his assistant Johannes Kepler (1571–1630) abandoned the theory of circular orbit for planets and proposed the elliptical orbits for planetary motion. He formulated three laws known as Kepler's laws of planetary motion.

KEPLER'S LAWS OF PLANETARY MOTION

Kepler at the Royal observatory at Prague succeeded, after 22 years of ceaseless work, in evolving the famous three laws, known after him—the first two in the year 1609 and the third in 1619. The three laws may be stated as follows:

First law (The law of orbits)

The orbit of a planet is an ellipse with sun at one of its foci.

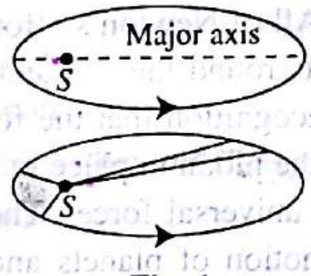


Fig. 1

Second law (The law of areas)

The line joining a planet to the sun sweeps out equal areas in equal intervals of times, i.e., the areal velocity of the planet is a constant.

This means that the planet moves more rapidly when it is closer to the sun than it does when it is far away

$$\text{i.e., } (dA/dt) = \text{a constant}$$

Third law (The law of periods or The harmonic law)

The square of the time period T of revolution of a planet around the sun is directly proportional to the cube of its semimajor axis a.

$$\text{i.e., } T^2 \propto a^3$$

For planets like Mercury, Venus and Earth with nearly cicrular orbits of radius r,

$$T^2 \propto r^3 \quad \therefore T^2/r^3 = k, \text{ a constant for all planets}$$

$$k = 10^{-13}\, s^2/m^3 = 1.33 \times 10^{-14}\ \text{day}^2/\text{kilometre}^3 (d^2\, km^{-3})$$

Proof of the second law

According to the geometrical meaning of angular momentum, the angular momentum of a planet of mass m orbiting around sun is

$$L = 2\text{ m} \times \text{ areal velocity}$$

$\therefore$ Areal velocity$= L/2m$ (i)

But $\tau = (dL/dt) = \vec{r} \times \vec{F} = 0 (\because \theta = 0)$ (ii)

Hence angular momentum L is a constant

$\therefore$ Areal velocity $= L/2m =$ a constant.

Thus Kepler's second law is prooved.

Proof of the third law

Assuming the orbit of a planet is a circle of radius r with Sun at the centre and T the period of the planet, then

$$GMm/r^2 = mr\omega^2 = mr(2\pi/T)^2;$$

where M is the mass of the sun, m the mass of the planet and T the period of revolution of the planet.

$\therefore T^2 = (4\pi^2/GM)r^3 = kr^3$; where $(4\pi^2/GM) = k$, a constant. $\therefore T^2 \propto r^3$

This is Kepler's third law

All of Newton's efforts in mechanics were directed towards explaining the motion of planets round the sun and of the moon round the earth. His supreme achievement lies in the recognition that the force that caused an apple to fall to the ground and the force that kept the moon in place in its orbit round the earth were only different manifestations of the same universal force. The form of this force necessary to explain the known facts about the motion of planets and the moon was worked out logically and his law of universal gravitation was formulated.

Example

VII.1. The distance of the moon from the earth is 3.84 × 10⁸ m and its time period is 27.3 days. Obtain the mass of the earth [NCERT]

The moon is a satellite of earth. Hence, if M is the mass of the earth and r is the distance of the moon from the earth, then according Kepler's third law

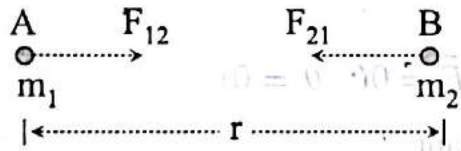
$$T^2 \propto r^3,\ \text{ie., } T^2 = kr^3 = (4\pi^2/GM)r^3 \quad \therefore M = 4\pi^2 r^3/GT^2$$

$$\therefore M = 4 \times 3.14^2 \times (3.84 \times 10^8)^3/[6.67 \times 10^{-11}(27.3 \times 24 \times 60 \times 60)^2]$$

$$= \mathbf{6.02 \times 10^{24}\ kg}$$

NEWTON'S LAW OF UNIVERSAL GRAVITATION

Every particle of matter in the universe attracts every other particle with a force that varies directly with the product of the masses and inversely with the square of the distance between them.



$F_{12} = F_{21} = F$

Fig. 2

If m_1 and m_2 are the masses of two particles separated by a distance r, the force of attraction between them is given by,

$$F \,\alpha\, \frac{m_1 m_2}{r^2};\quad F = G\frac{m_1 m_2}{r^2};$$

where G is called **Gravitational Constant.** If $m_1 = m_2 = 1$ and $r = 1$; then $G = F$. Hence, **Gravitational Constant** *is numerically equal to the force of attraction between two unit masses separated by unit distance.*

$$\text{Dimensions of G} = \frac{Fr^2}{m_1 m_2} = \frac{MLT^{-2} \times L^2}{M^2} = M^{-1}L^3T^{-2}$$

In SI units, $G = 6.67 \times 10^{-11}\ \text{Nm}^2\,\text{kg}^{-2}$

Vector form of Newton's law of gravitation

Let m_1 and m_2 be two particles situated at a distance r apart. Gravitational force exerted on A by B, $\vec{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{r}_{21}$. Here *negative sign* indicates that direction of $\vec{F}_{12}$ is opposite to that of $\hat{r}_{21}$ and it shows *that the force is attractive in nature.*

Regarding gravitational force it is worth noting that:–

1. It is always attractive
2. It is independent of the intervening medium
3. It is an action–reaction pair
4. It is a two body interaction i.e. gravitational force between bodies is independent of the presence or absence of other bodies
5. The gravitational force is a central force as it acts along the line joining the centres of the two bodies
6. The gravitational force is conservative
7. It holds good over wide range of distances. It is found to be true from interplanetary distance to inter-atomic distance.

Examples

VII.2. Two bodies of mass 10 kg and 25 kg are placed at a distance of 0.2 m apart. Find the force between them. $G = 6.67 \times 10^{-11}$ SI units.

$$m_1 = 10\,\text{kg}; \quad m_2 = 25\,\text{kg}; \quad r = 0.2\,\text{m}$$

$$F = G\frac{m_1 m_2}{r^2} = \frac{6.67 \times 10^{-11} \times 10 \times 25}{0.2^2} = \mathbf{4.169 \times 10^{-7}\ N}$$

VII.3. A mass M is split into two parts m and $(M - m)$ which are separated by a certain distance. What ratio m/M maximises the gravitational force between the parts?

If r is the distance between m and $(M - m)$ the gravitational force will be

$$F = \frac{Gm(M-m)}{r^2} = \frac{G}{r^2}(mM - m^2)$$

For F to be maximum $\frac{dF}{dm} = 0$ and r is constant

$$\frac{G}{r^2} \times \frac{d}{dm}(mM - m^2) = 0 \therefore \frac{d}{dm}(mM - m^2) = 0,$$

$$M - 2m = 0 \qquad \frac{m}{M} = \frac{1}{2}$$

The force will be maximum when the parts have equal mass.

g = 9.8
G = 6.67×

VII.4. **A rocket is fired towards the sun. At what point on the path is the gravitational force on the rocket zero. Mass of the sun = 2 × 10^{30} kg, Mass of the earth = 6 × 10^{24} kg, orbital radius of earth = 1.5 × 10^{11} m. Ignore the presence of other satellites.** **[NCERT]**

Let x be the distance from the earth where gravitation force on the rocket is zero.

$$\therefore \frac{GM_e m_r}{x^2} = \frac{GM_s m_r}{(r-x)^2};$$

where M_e, M_s and m_r are the masses of the earth, sun and rocket respectively.

$$\frac{M_e}{x^2} = \frac{M_s}{(r-x)^2}$$

$$\frac{6 \times 10^{24}}{x^2} = \frac{2 \times 10^{30}}{(1.5 \times 10^{11} - x)^2};$$

$$\left(\frac{x}{1.5 \times 10^{11} - x}\right)^2 = 3 \times 10^{-6}$$

$$\frac{x}{1.5 \times 10^{11} - x} = 10^{-3}\sqrt{3};$$

$$10^3 x = \sqrt{3} \times 1.5 \times 10^{11} - \sqrt{3}x$$

$$x(10^3 + \sqrt{3}) = \sqrt{3} \times 1.5 \times 10^{11};$$

$$x = \frac{\sqrt{3} \times 1.5 \times 10^{11}}{10^3 + \sqrt{3}} = \mathbf{2.6 \times 10^8\ m}$$

Acceleration due to gravity g

The uniform acceleration produced in a freely falling body due to gravitational pull of the earth is known as acceleration due to gravity.

Let us consider earth to be a perfect sphere of mass M and radius R. Consider a body of mass m placed on the surface of the earth. Assuming the mass of the earth to be concentrated at its centre, the force of gravity or the weight of the body is given by,

$$W = \frac{GMm}{R^2} = mg; \quad g = \frac{GM}{R^2}$$

Clearly 'g' is independent of the mass m of the body.

Mass and Density of the earth

Since, $g = \dfrac{GM}{R^2}$, the mass of the earth, $M = \dfrac{gR^2}{G}$

Assuming the earth to be a homogeneous sphere,

Volume of the sphere $= \frac{4}{3}\pi R^3$

Density of the earth, $= D = \dfrac{M}{V} = \dfrac{gR^2 \times 3}{G \times 4\pi R^3} = \mathbf{3g/4\pi RG}$

Variation in the acceleration due to gravity of the earth

1. Variation of 'g' due to the shape of the earth

The value of acceleration due to gravity on the surface of the earth is $g = \frac{GM}{R^2}$ where G and M are constants.

$$\text{Hence,} \quad g \,\alpha\, \frac{1}{R^2}$$

Again the polar radius is less than the equatorial radius of the earth; therefore the value of g is more at the poles than at the equator.

2. Variation of g with Latitude. Rotation of the Earth

Consider a body of mass m located on the surface of the earth at a latitude λ. Due to rotation of the earth about its axis this body experiences a centrifugal force F_{cf} given by

$$F_{cf} = mR\cos\lambda \times \omega^2$$

where R is the radius of the earth and ω the angular velocity of the earth. This force is directed away from the axis of rotation.

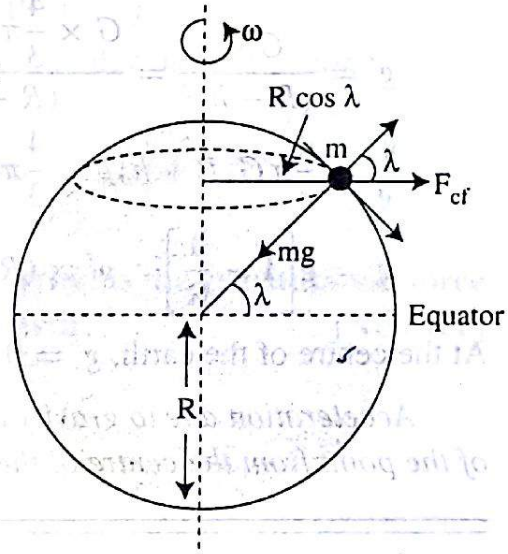


Fig. 3

The resultant force on the mass towards the centre of the earth

$$mg' = mg - F_{cf}\cos\lambda = mg - mR\cos\lambda \cdot \omega^2 \cos\lambda$$

$$= m(g - R\omega^2\cos^2\lambda)$$

$$\text{or} \quad g' = g - R\omega^2\cos^2\lambda$$

At the equator $\lambda = 0$ $\quad \therefore g' = g - R\omega^2$

At the poles $\lambda = 90°$ $\quad \therefore g' = g$

3. Variation of g with altitude

Let g be the acceleration due to gravity on the surface of the earth and g' the acceleration due to gravity at a height h above the earth's surface. Assuming earth to be a perfect sphere,

$$g = \frac{GM}{R^2} \quad \text{and} \quad g' = \frac{GM}{(R+h)^2}$$

$$\frac{g}{g'} = \left(\frac{R+h}{R}\right)^2 = \left(1 + \frac{h}{R}\right)^2$$

$$g' = \frac{g}{(1+h/R)^2} = g\left(1+\frac{h}{R}\right)^{-2} = g\left(1 - \frac{2h}{R}\right) \quad \text{approximately,}$$

Thus acceleration due to gravity decreases with altitude.

4. Effect of depth on the value of g–variation of g with depth

Consider earth to be a homogeneous sphere of radius R and mass M with centre O. If g is the acceleration due to gravity on the surface of the earth

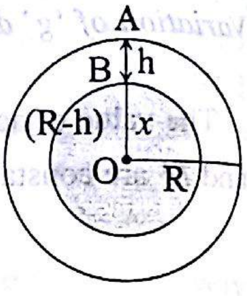


Fig. 4

$$g = \frac{GM}{R^2} = \frac{G \times \frac{4}{3}\pi R^3}{R^2}\rho = \frac{4}{3}\pi G R\rho;$$

where ρ is the mean density of the earth.

Let g' be the acceleration due to gravity at a depth h below the surface of the earth, then,

$$g' = \frac{GM'}{(R-h)^2} = \frac{G \times \frac{4}{3}\pi (R-h)^3}{(R-h)^2}\rho = \frac{4}{3}\pi G(R-h)\rho.$$

$$\frac{g'}{g} = \frac{4}{3}\pi G(R-h)\rho \div \frac{4}{3}\pi G R\rho = \frac{R-h}{R}$$

$$\mathbf{g' = g\left[1 - \frac{h}{R}\right]} \quad \therefore g' \propto (R-h) \text{ or } g' \propto x.$$

At the centre of the earth, $g' = 0$ (since $h = R$)

Acceleration due to gravity at any point inside the earth is proportional to the distance of the point from the centre of the earth.

Examples

VII.5. Calculate the mass of the earth from the following data. Radius of the earth = 6371 km. Acceleration due to gravity = 9.8 ms^{-2}. $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}

$$g = \frac{GM}{R^2}; \quad M = \frac{gR^2}{G} = \frac{9.8 \times (6371 \times 10^3)^2}{6.67 \times 10^{-11}} = \mathbf{6 \times 10^{24}\ kg.}$$

VII.6. The acceleration due to gravity at the moon's surface is 1.67 ms^{-2}. If the radius of the moon is 1.74×10^6 m, calculate the mass of the Moon. $G = 6.67 \times 10^{-11}$ SI unit

$$g = \frac{GM}{R^2}; M = \frac{gR^2}{G}$$

$$M = \frac{1.67 \times (1.74 \times 10^6)^2}{6.67 \times 10^{-11}} = \mathbf{7.58 \times 10^{22}\ kg}$$

VII.7. At what height above the earth's surface the value of g is the same as in a mine 100 km deep?

$$g' = g\left[1 - \frac{2h}{R}\right] = g\left[1 - \frac{h'}{R}\right]$$

$$\frac{2h}{R} = \frac{h'}{R} \qquad h = \frac{h'}{2} = \frac{100}{2} = \mathbf{50\ km}$$

VII.8. How much below the surface does the acceleration due to gravity become 70% of its value on the surface of the earth? $R = 6.4 \times 10^6$ m.

$$\frac{g'}{g} = 1 - \frac{h}{R}$$

$$\frac{g'}{g} = \frac{70}{100}; \quad \frac{7}{10} = 1 - \frac{h}{R}; \quad \frac{h}{R} = \frac{3}{10}; \quad h = \frac{3R}{10} = \frac{3 \times 6.4 \times 10^6}{10}$$

$$= \mathbf{1.92 \times 10^6\ m}$$

VII.9. If earth were a perfect sphere of radius 6.37×10^6 m, rotating about its axis with a period of one day ($= 8.64 \times 10^4$ s) how much would the acceleration due to gravity differ from pole to equator? **[NCERT]**

$$R = 6.37 \times 10^6 \text{ m}; \quad T = 8.64 \times 10^4 \text{ s}; \quad g - g' = ?$$

If g is the acceleration due to gravity at the pole and g' that at the equator,

$$g' = g - R\omega^2 \cos^2 \lambda = g - R\omega^2 (\lambda = 0, \text{ along the equator})$$

$$\therefore g - g' = R\omega^2 = R(2\pi/T)^2 = 6.37 \times 10^6 (2\pi/8.64 \times 10^4)^2$$

$$= \mathbf{3.37 \times 10^{-2}\ ms^{-2}}$$

VII.10. A body weighs 63 N on the surface of the earth. What is the gravitational force on it at a height equal to half the radius of the earth? **[NCERT]**

On the surface of the earth, $g = \dfrac{GM}{R^2}$

At a height h above the earth, $g' = \dfrac{GM}{(R+h)^2}$

$$\frac{g'}{g} = \frac{R^2}{(R+h)^2} = \frac{R^2}{(R+R/2)^2} = \frac{4}{9}; \quad g' = \frac{4g}{9}$$

Gravitational force at the height,

$$mg' = m \times \frac{4g}{9} = mg \times \frac{4}{9} = 63 \times \frac{4}{9} = \mathbf{28\,N}$$

VII.11. Imagine a tunnel dug along a diameter of the earth. Show that a particle dropped from one end of the tunnel executes simple harmonic motion. What is the time period? **[NCERT]**

[*Attempt after learning chapter XII, the oscillations*]

When the particle is at a depth h below the surface its displacement from the centre $x = (R - h)$; where R is the radius of the earth.

Acceleration due to gravity at a depth h below the surface of the earth,

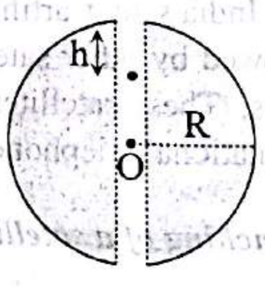


Fig. 5

$$g' = g(1 - h/R) = \frac{g(R-h)}{R} = \frac{gx}{R};$$

where x is the displacement of the particle from the centre. Here acceleration is proportional to displacement and directed towards the centre.

Hence the particle oscillates simple harmonically. Its period is given by,

$$T = 2\pi\sqrt{R/g} = 2 \times 3.14\sqrt{\frac{6.4 \times 10^6}{9.8}} = \mathbf{5075\,s = 1h\,25\,min}$$

VII.12. The acceleration due to gravity on the moon's surface is $1.7\,ms^{-2}$. The radius of the moon is 0.27 times that of the earth. Acceleration due to gravity on the surface of the earth is $9.8\,ms^{-2}$. Find the ratio of the mass of the earth to that of the moon.

For the earth $g_e = \frac{GM_e}{R_e^2}$ (1)

For the moon $g_m = \frac{GM_m}{R_m^2}$ (2)

$$\therefore \frac{g_e}{g_m} = \frac{M_e}{M_m} \times \left(\frac{R_m}{R_e}\right)^2$$

$$\frac{M_e}{M_m} = \left(\frac{g_e}{g_m}\right) \times \left(\frac{R_e}{R_m}\right)^2 = \frac{9.8}{1.7} \times \left(\frac{1}{0.27}\right)^2 = 77.27$$

SATELLITES

A satellite is a body which is constantly revolving in an orbit around a planet. Some planets of the solar system have natural satellites orbiting them. The moon is the natural satellite of the earth. It revolves round the earth once in 27.3 days.

Artificial Satellites

It is a man-made object placed at a height above the earth and given sufficient velocity so as to revolve round the earth in closed orbit.

The first artificial satellite Sputnik–1 was launched into orbit round the earth by Russian scientists on October 4, 1957. Since then many countries have joined the race for space exploration. Artificial satellites launched by many countries are currently revolving the earth in orbits at a height of hundreds of kilometres.

India's first artificial satellite *Aryabatta* was launched on April 19, 1975. This was followed by other satellites. India has also put into orbit communication satellites *INSAT* series. These satellites are multifunction satellites which relay long distance national and international telephone communications, television signals, weather data etc.

Launching of a satellite

To launch a satellite into orbit multistage rockets are used. These rockets generally have three stages. The satellite is kept at the top of the third stage. The first part of the flight of the rocket is made vertical so that it may cross the denser part of the atmosphere quickly. As the first stage rocket fuel burns off, this stage drops and the second stage begins to burn. At this stage it is given a gradual tilt. Since the mass is considerably reduced and there is less resistance of air, the velocity of the system increases considerably. When the fuel of the second stage is finished it is detached and the third stage gets ignited. The final stage rocket turns the satellite in a horizontal direction, gives it the proper speed and lauches it in space which makes the satellite revolve in closed orbits round the earth.

Orbital velocity

The velocity with which a satellite moves in its closed orbit is called orbital velocity.

For a satellite to be in stable orbit it must have a suitable velocity, which depends on the radius of the orbit.

Consider a satellite of mass m moving round in a closed orbit of radius r with orbital velocity v. Let M be the mass of the earth and R its radius.

When the satellite is in stable orbit, the centripetal force is provided by the gravitational force,

$$i.e., \frac{mv^2}{r} = \frac{GMm}{r^2}; \quad v = \sqrt{\frac{GM}{r}}$$

The orbital velocity is inversely proportional to the square root of the radius of the orbit.

If h is the height of the satellite above the earth, $r = R + h$

$$v = \sqrt{\frac{GM}{R+h}}$$

$$\text{But } g = \frac{GM}{R^2}; \quad GM = R^2 g; \quad v = \sqrt{R^2 g/(R+h)}$$

Minimum orbit – First Cosmic Velocity

If the orbit is close to the earth, h is negligible compared to R, then orbital velocity

$$v = \sqrt{\frac{GM}{R}} = \sqrt{Rg}$$

This orbit is called *minimum orbit.* The velocity corresponding to minimum orbit is called *first cosmic velocity*

First cosmic velocity, $v = \sqrt{Rg} = \sqrt{6.4 \times 10^6 \times 9.8} = 7920\,ms^{-1} = \mathbf{7.92\,kms^{-1}}$

Time period of a satellite

It is the time taken by the satellite to revolve once round the earth. If r is the radius of the orbit and v the orbital velocity, time period,

$$T = \frac{2\pi r}{v}; \text{ But, } v = \sqrt{\frac{GM}{r}} \therefore T = \frac{2\pi r}{\sqrt{GM/r}} = 2\pi\sqrt{\frac{r^3}{GM}}$$

For minimum orbit, $T = 2\pi\sqrt{R^3/GM} = 2\pi\sqrt{R/g}$; $\left[\text{Since, } g = \frac{GM}{R^2}\right]$

Geostationary satellites (Geosynchronous satellites)

They are artificial satellites orbiting the earth so that their period is synchronous (same) with that of the earth. The orbit in which they are staying is called *parking orbit* or *geo-stationary orbit*. Such satellites must move in circular orbits in the equatorial plane of the earth and in the same direction as the rotation of the earth i.e. from west to east. Its angular velocity will also be the same as that of the earth.

Period of a synchronous satellite is $T = 24$ hr; its angular velocity

$$\omega = 2\pi/T = 2\pi/(24 \times 60 \times 60) \text{ rad } s^{-1}$$

It orbits around the earth at a height of about 36,000 km.

Such satellites are used for communication purpose, and are called SYNCOMS [Synchronous Communication Satellites]. Radio, TV and telephone signals are directed to a satellite from a station on the earth and the satellite retransmits them to some other station on earth. In this way the range of communication has been increased. Accurate and precise weather forecasting can be made with the help of artificial satellites.

POLAR SATELLITE

Polar orbiting satellites orbit closely parallel the earth's meridian lines. They pass over the north and south poles in each revolution. As the earth rotates to the east beneath the satellite, each pass monitors an area to the west of the previous pass. These 'strips' can be pieced together to produce a picture of a larger area.

Polar satellites have the advantage of photographing clouds directly beneath them. Geostationary satellite images of the polar regions are distorted because of the low angle the satellite sees the region. Polar satellites also circle at a much lower altitude (about 850 km). The satellite orbit is inclined at about 99° relative to the Equator so that the satellite passes near the North and South Poles. The direction of travel is south to north during one half of the orbit (called *ascending leg*) and north to south during the other half (the *descending leg*). Each orbit takes about 100 minutes, resulting in about 14 orbits a day. The satellite covers the entire surface of the Earth twice each day. As it travels around the Earth, a polar-orbiting satellite continuously scans the atmosphere and surface of the Earth below it and concurrently transmits the data it collects back to Earth.

One purpose of Polar satellite is to gather information that will help scientists protect future satellites from radiation and other atmospheric dangers. Since the satellite is flying in the upper atmosphere, there is some concern that the satellite's instruments may be affected in some way or damaged by the sun's harmful radiation. One of the main atmospheric studies of the Polar satellite is to observe ions in the Earth's atmosphere, especially in the polar regions. These observations include measurements of the partial pressure, wind velocity, and temperature of the ions. These ions include hydrogen, helium, nitrogen, oxygen, and molecules of these species and their compounds. These ionised gases make up what scientists call the plasma environment. Relatively low orbit allows detection and collection of data, by instruments aboard a polar-orbiting satellite, at a higher spatial resolution than from a geostationary satellite. The high resolution data combined with global coverage allows polar-orbiting systems to provide real-time environmental information for initializing Global Forecast Models and improving output accuracy.

Examples

VII.13. Find the orbital velocity of an artificial satellite of the earth ($g = 9.8\,ms^{-2}$; Radius of the earth = 6400 km)

Since the height of the satellite is not given, we find the first cosmic velocity.

$$\text{Orbital velocity } v = \sqrt{gR} = \sqrt{9.8 \times 64 \times 10^5} = 7920\,\text{ms}^{-1} = \mathbf{7.92\,km\,s^{-1}}$$

VII.14. An artificial satellite is revolving round the earth at a height of 30 km above the surface of the earth. Find the orbital velocity. ($R = 6370$ km; $g = 9.8\,\text{ms}^{-2}$)

$$R = 6370 \times 10^3\,\text{m}; \quad h = 30 \times 10^3\,\text{m}; \quad g = 9,8\,\text{ms}^{-2}; \quad v = ?$$

$$v = \sqrt{R^2 g/(R+h)} = \sqrt{(6370 \times 10^3)^2 \times 9.8/6400 \times 10^3}$$

$$= 7882\,\text{ms}^{-1} = \mathbf{7.882\,km/s}$$

VII.15. An earth satellite makes a complete circuit around the earth in 90 minutes. If the orbit is circular, calculate the height of the satellite above the earth. Radius of the earth = 6370 km; $g = 9.8\,\text{ms}^{-2}$.

$$T = 90' = 90 \times 60 = 5400\,s; \; R = 6370\,km = 6370 \times 10^3\,m; \; g = 9.8\,\text{ms}^{-2}; \; h = ?$$

$$\frac{GMm}{r^2} = mr\omega^2 = mr\left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2 mr}{T^2}$$

$$r^3 = \frac{GMT^2}{4\pi^2}; r = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3} = \left[\frac{gR^2T^2}{4\pi^2}\right]^{1/3}$$

$$r = \left[\frac{9.8 \times (6370 \times 10^3)^2 \times 5400^2}{4 \times \pi^2}\right]^{1/3} = 6649 \times 10^3\,\text{m} = 6649\,\text{km}$$

$$r = R + h; h = r - R = 6649 - 6370 = \mathbf{279\,km}$$

VII.16. An artificial satellite of mass 100 kg is in a circular orbit at 500 km above earth's surface. Taking the radius of the earth as 6.5×10^6 m find (a) the acceleration due to gravity at any point along the satellite path. (b) centripetal acceleration of the satellite?

$$h = 500\,\text{km}; \; R + h = 6.5 \times 10^6 + 0.5 \times 10^6 = 7 \times 10^6\,m; \quad g' = ?; \; a = ?$$

$$(a) \quad g' = g\left[\frac{R}{R+h}\right]^2 = 9.8 \times \left(\frac{6.5 \times 10^6}{7 \times 10^6}\right)^2 = \mathbf{8.45\,ms^{-2}}$$

$$(b) \quad a = \frac{v^2}{r} = \frac{gR^2/(R+h)}{(R+h)} = R^2 g/(R+h)^2$$

$$= (6.5 \times 10^6)^2 \times 9.8/(7 \times 10^6)^2 = \mathbf{8.45\,ms^{-2}}$$

VII.17. Calculate the height above the earth at which the geostationary satellite is orbiting the earth. Radius of the earth 6400×10^3 m; mass of earth 6×10^{24} kg.

Period of satellite = Period of earth = 24 hours = 86400 s

Let r be the radius of the orbit and m the mass of the satellite.

$$\frac{GMm}{r^2} = mr\omega^2 = mr\left(\frac{2\pi}{T}\right)^2, r^3 = \frac{GMT^2}{4\pi^2}, \quad r = \left(\frac{GMT^2}{4\pi^2}\right)^{\frac{1}{3}}$$

$$r = \left[\frac{6.67 \times 10^{-11} \times 6 \times 10^{24} \times 86400^2}{4\pi^2}\right]^{\frac{1}{3}}$$

$$= 4.2312 \times 10^7 \text{ m} = 42312 \text{ km}$$

Height of the satellite above the earth = 42312 − 6400 = **35912 km**

VII.18. To what latitude does the *symcoms* coverage extend? What is the orbital speed of the *symcoms*? (Orbital radius of the SYMCOMS = 4.22 × 10^4 km; Radius of the earth $R = 6.37 \times 10^3$ km) [NCERT]

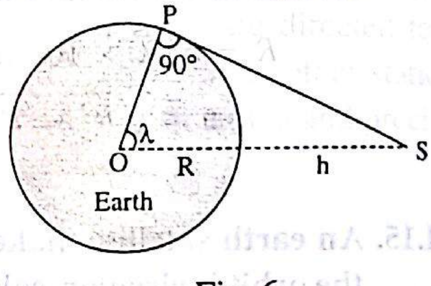


Fig. 6

Let s be the satellite and p be a place on the earth at a latitude λ.
From the figure it is clear that the latitude of the coverage extends up to the tangent SP

$$\text{Cos}\lambda = OP/OS = 6.37 \times 10^3/4.22 \times 10^4 = 0.151 \quad \therefore \lambda \quad = \mathbf{81.3°}$$

Note: About a 9° circular arc around the pole is left uncovered. Thus we need three satellites to cover the entire earth.
Orbital speed of the satellite

$$= 2\pi r/T = 2 \times 3.14 \times 4.22 \times 10^7/24 \times 60 \times 60 = \mathbf{3.07 \times 10^3\ ms^{-1}}$$

VII.19. A satellite of the earth revolves in circular orbits at a height 250 km above the earth's surface. What is the period of revolution of the satellite? $R = 6.38 \times 10^6$ m; $g = 9.8$ ms^{-2}

$$v = \sqrt{R^2 g/r}, \quad T = \frac{2\pi r}{v} = \frac{2\pi r}{R}\left(\frac{r}{g}\right)^{\frac{1}{2}} = \frac{2\pi}{R}\left(\frac{r^3}{g}\right)^{\frac{1}{2}}$$

$$R = 6.38 \times 10^6 \text{ m}; \quad h = 0.25 \times 10^6 \text{ m}, \ r = R + h = 6.63 \times 10^6 \text{ m}$$

$$\therefore \quad T = \frac{2\pi}{6.38 \times 10^6}\left[\frac{(6.63 \times 10^6)^3}{9.8}\right]^{\frac{1}{2}} = \mathbf{5365\ s}$$

VII.20. The mean orbital radius of the earth around the sun is 1.5 × 10^8 km. The period of revolution of the earth is 365 days. $G = 6.7 \times 10^{-11}$ Nm^2 kg^{-2}. Estimate the mass of the sun. [NCERT]

$$\frac{GM_s m_e}{r^2} = m_e r\omega^2 = m_e r\left(\frac{2\pi}{T}\right)^2$$

$$M_s = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2(1.5 \times 10^{11})^3}{6.7 \times 10^{-11}(365 \times 24 \times 60 \times 60)^2}$$

$$= \mathbf{2 \times 10^{30}\ kg}$$

GRAVITATIONAL FIELD

Gravitational field of a body is the space around the body where its gravitational influence is felt.

Intensity of gravitational field (I)

The intensity of gravitational field at a point is defined as the force experienced by a body of unit mass placed at that point.

*Gravitational intensity at a point due to a point mass

Consider a point P distant r from a body of mass M. Let a test-mass m be placed at P.

Fig. 7

Gravitational force on mass m, $F = \frac{GMm}{r^2}$

Gravitational intensity at P = Force on unit mass $= \frac{GM}{r^2}$

Gravitational field of the earth

The gravitation force on a mass m placed near the surface of the earth is given by, $F = \frac{GMm}{R^2}$; where M is the mass and R the radius of the earth.

Intensity of gravitational field, $I = \frac{F}{m} = \frac{GM}{R^2} = g$, the acceleration due to gravity.

Thus the intensity of gravitational field near the surface of the earth is equal to the acceleration due to gravity at the place.

GRAVITATIONAL POTENTIAL

In order to move a mass in a gravitational field work has to be done in overcoming gravitational force of attraction.

Gravitational potential at a point (V)

It is defined as the work done in bringing unit mass from infinity to that point. It is also measured as the potential energy per unit mass of a body placed at that point.

The potential corresponds to the level in the case of flow of liquids or temperature in the case of heat transfer. Hence a body in a gravitational field moves from a higher potential to a lower potential. Gravitational potential at infinite distance from a point mass is zero, as the gravitational intensity at that point is zero. The maximum value of potential is zero and is at infinity with respect to the given mass. The potential decreases as we approach the point mass. *Hence gravitational potential at any point in the gravitational field of a body is negative.*

***Potential at a point due to a point mass**

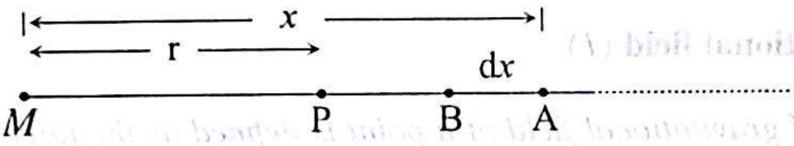


Fig. 8

P is a point distant r from a point mass M. Consider a point A, distant x from M.

Gravitational field intensity at $A = \dfrac{GM}{x^2}$ = Force on unit mass placed at A

Work done in moving unit mass from A through a small distance dx is

$$dW = F \times dx = \frac{GM \times dx}{x^2}$$

Total work done in moving unit mass from infinity to P,

$$W = \int_{\infty}^{r} \frac{GM \times dx}{x^2} = \left[\frac{-GM}{x}\right]_{\infty}^{r} = \frac{-GM}{r};$$

$\therefore$Potential at P, $V = \dfrac{-GM}{r}$

Gravitational potential due to a sphere

Consider a spheres of radius R and mass M. Potential at a point distant $r (r > R)$ from the centre of the sphere is given by

$$V_p = -\frac{GM}{r}$$

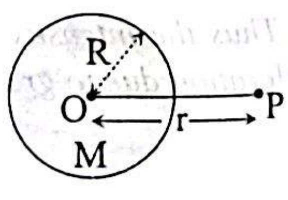


Fig. 9

A body of mass m placed at that point possesses potential energy,

$$E_p = -\frac{GMm}{r}$$

Gravitational potential energy of the earth

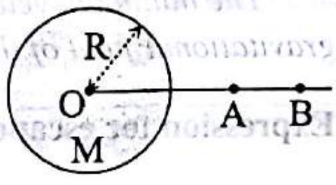


Fig. 10

Let M be the mass of the earth and R its radius. Consider two points A and B distant r_1 and r_2 from the centre of the earth $(r_1, r_2 > R)$

$$\text{Potential energy of a mass } m \text{ placed at } A = -\frac{GMm}{r_1}$$

$$\text{Potential energy when it is place at } B = -\frac{GMm}{r_2}$$

$$\text{Increase in P.E. when it is moved from } A \text{ to } B = -\frac{GMm}{r_2} - \left(-\frac{GMm}{r_1}\right)$$

$$= GMm\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

If the surface of the earth is taken as zero potential energy level, the potential energy of the mass at a height h is obtained by putting $r_1 = R$ and $r_2 = R + h$

P.E of the mass with respect to the surface of the earth,

$$= GMm\left[\frac{1}{R} - \frac{1}{R+h}\right] = GMm\left[\frac{h}{R(R+h)}\right]$$

If $h \ll R$, then, $\text{P.E} = \frac{GMmh}{R^2} = mgh$; (Since $\frac{GM}{R^2} = g$)

Total energy of an orbiting satellite

Let M be the mass of the earth and m the mass of the satellite. Let r be the radius of the orbit of the satellite and v the velocity of the satellite

$$\text{Potential energy of the satellite} = -GMm/r$$

$$\text{Kinetic energy of the satellite} = (1/2)mv^2$$

From the law of gravitation

$$mv^2/r = GMm/r^2; \qquad \text{KE} = (1/2)mv^2 = +GMm/2r$$

$$\text{Total energy of the orbiting satellite} = \text{P.E.} + \text{K.E.}$$

$$= -GMm/r + GMm/2r = -GMm/2r$$

$$\text{Binding energy of the satellite} = +GMm/2r$$

The kinetic and total energy are equal in magnitude. The total energy is negative and half the potential energy

Escape speed of a body from the earth

If a body is thrown vertically upwards it rises up to a certain height and then returns. The height to which a body rises depends on the velocity of projection. If a body is given sufficient kinetic energy to overcome the gravitational pull of the earth it will escape into the outer space and never return.

The escape velocity from the earth is called the **second cosmic velocity**.

The minimum velocity with which a body must be projected so that it may escape from gravitational field of the earth is called escape velocity from the earth.

Expression for escape speed (v_e)

Let M be the mass of the earth and R its radius. Let v_e be the velocity of a body of mass m with which it is to be projected so that it escapes the gravitational field of the earth.

$$\text{KE of the body near the surface of the earth} = \frac{1}{2}mv_e^2$$

$$\text{PE of the body on the surface of the earth} = -\frac{GMm}{R}$$

$$\text{Total energy of the body near the surface} = \frac{1}{2}mv_e^2 - \frac{GMm}{R}$$

At infinite distance both K.E and P.E are zero. Applying the law of conservation of energy,

$$\frac{1}{2}mv_e^2 - \frac{GMm}{R} = 0; \quad \frac{1}{2}mv_e^2 = \frac{GMm}{R}$$

$$v_e = \sqrt{\frac{2GM}{R}}; \quad v_e = \sqrt{2Rg}, \ \text{since } g = \frac{GM}{R^2}$$

Escape velocity from the earth $= 11.2\,\text{kms}^{-1}$

Note: The escape speed does not depend on the direction in which the projectile is fired. However, the rotation of earth does play a role. Firing eastward has an advantage in that the earth's tangential speed (about 0.46 km/s) can be subtracted from the calculated value of v_e(11.2 km/s).

Weightlessness in orbit

Weight of a body is the reaction it exerts on its support. If a person of mass m is standing inside a lift the reaction he exerts on the floor of the lift is $R = mg$. When the lift moves down with an acceleration a, the reaction, $R = m(g - a)$. If the suspension cable of the lift snaps it has a free fall and in this case $a = g$. The reaction $R = m(g - g) = 0$. The person feels apparently weightless.

A similar situation prevails in the case of an orbiting satellite. To understand what is happening let us consider a projectile which is shot parallel to the earth's surface, at a height sufficient to allow us to neglect the effect of atmosphere. The projectile is caused to fall to the earth by the pull of gravity. If the speed of the projectile is large enough the freely falling object will simply circle the earth. Although the projectile is falling continuously towards earths centre, the curvature of the earth is the same as the curvature of the projectiles path. The speed of the projectile on in its circular path is just proper so that this required centripetal acceleration is equal to the gravitational acceleration provided by the earth.

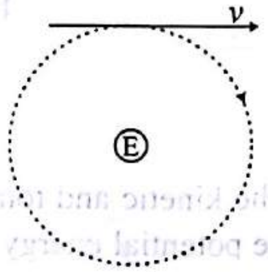


Fig. 11

The spaceship is also a projectile shot horizontally. As it circles the earth it is a freely falling object. Everything within it is accelerated towards the centre of the earth with acceleration due to gravity. Hence no supporting force is needed for objects within the space ship. Hence all objects in the spaceship appear weightless.

Examples

VII.21. Determine the escape velocity of a body from the moon. Take the moon to be a uniform sphere of radius 1.74×10^6 m and mass 7.36×10^{22} kg.

$R = 1.74 \times 10^6\, m$; $M = 7.36 \times 10^{22}$ kg $G = 6.67 \times 10^{-11}$ S.I.unit; $v_e = ?$

$$v_e = \sqrt{2GM/R} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 7.36 \times 10^{22}}{1.74 \times 10^6}}$$

$$= \mathbf{2375\, ms^{-1} = 2.375\, km/s}$$

VII.22. Jupiter has mass 318 times that of the earth and its radius is 11.2 times the earth's radius. Estimate the escape velocity of a body from Jupiter surface, given that the escape velocity from earth's surface is $11.2\, km\, s^{-1}$.

Escape velocity on the surface of earth,

$$v_e = \sqrt{\frac{2GM_e}{R_e}} = 11.2\, kms^{-1}$$

$$\text{Escape velocity on Jupiter } v_J = \sqrt{\frac{2GM_J}{R_J}}$$

$$= \sqrt{\frac{2 \times G \times 318 \times M_e}{11.2 \times R_e}} = \sqrt{2GM_e/R_e} \times \sqrt{\frac{318}{11.2}}$$

$$= 11.2 \times \sqrt{\frac{318}{11.2}} = \sqrt{11.2 \times 318} = \mathbf{59.7\, kms^{-1}}$$

VII.23. A satellite orbits the earth at a height of 500 km from its surface. Compute (a) kinetic energy (b) potential energy and (c) total energy of the satellite. [Mass of satellite = 300 kg, radius of the earth = 6.4×10^6 m; $G = 6.67 \times 10^{-11}\, Nm^2Kg^{-2}$; mass of earth = 6×10^{24} kg]

$$(a)\ K.E = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{GM}{(R+h)}$$

$$= \frac{1}{2} \times \frac{300 \times 6.67 \times 10^{-11} \times 6 \times 10^{24}}{6900 \times 10^3} = \mathbf{8.7 \times 10^9\, J}$$

$$(b)\ P.E = \frac{-GMm}{R+h} = \frac{-6.67 \times 10^{-11} \times 6 \times 10^{24} \times 300}{6900 \times 10^3} = \mathbf{-17.4 \times 10^9\, J}$$

$$(c)\ \text{Total energy} = P.E + K.E = -17.4 \times 10^9 + 8.7 \times 10^9 = \mathbf{-8.7 \times 10^9\, J}$$

VII.24. A rocket is fired vertically with a speed of $5\, km\, s^{-1}$ from the earth's surface. How far from the earth does the rocket go before returning to the earth? Mass of the earth = 6×10^{24} kg; mean radius of the earth = 6.4×10^6 m; [NCERT]

Let the rocket reach a height h where its velocity vanishes. If m is its mass, at the surface of the earth its total energy

$$= K.E + P.E = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

Its total energy at a height $h = 0 - \frac{GMm}{R+h} = -GMm/R+h$

According to the law of conservation of energy,

$$1/2mv^2 - GMm/R = -GMm/(R+h)$$

$$\therefore \frac{1}{2}mv^2 = \frac{GMm}{R} - \frac{GMm}{R+h}$$

$$\frac{1}{2}v^2 = \frac{GM}{R} - \frac{GM}{R+h} = GM\left(\frac{1}{R} - \frac{1}{R+h}\right) = \frac{GMh}{R(R+h)}$$

$$\therefore h = \frac{R^2v^2}{2GM - Rv^2}$$

$$\therefore \quad h = \frac{(6.4 \times 10^6)^2 \times (5 \times 10^3)^2}{(2 \times 6.67 \times 10^{-11} \times 6 \times 10^{24}) - (6.4 \times 10^6)(5 \times 10^3)^2}$$

$$= \mathbf{1.6 \times 10^6\ m}$$

VII.25. A satellite orbits the earth at a height of 400 km **above the surface. How much energy must be expended to rocket the satellite out of earth's gravitational influence. Mass of the satellite =** 200 kg; **mass of the earth =** 6×10^{24} kg; **radius of the earth =** 6.4×10^6 m; $G = 6.67 \times 10^{-11}$ **S.I. unit. [NCERT]**

Total energy of the satellite when it is put in to orbit.

$$= P.E + K.E = -\frac{GMm}{R+h} + \frac{1}{2}mv^2$$

$$= -\frac{GMm}{R+h} + \frac{1}{2}m\frac{GM}{R+h} \quad \left(\because v = \sqrt{\frac{GM}{R+h}}\right)$$

$$= \frac{-GMm}{2(R+h)} = \frac{-6.67 \times 10^{-11} \times 6 \times 10^{24} \times 200}{2 \times 6.8 \times 10^6} = \mathbf{-5.88 \times 10^9\ J}$$

Total energy of satellite out of earth's gravitational influence = 0

Energy required to make the satellite go out of earth's gravitational field

$= 0 - (5.88 \times 10^9) = \mathbf{5.88 \times 10^9\ J}$

VII.26. The escape velocity of a projectile on the earth's surface is $11.2\ \text{kms}^{-1}$. **A body is projected with twice this speed. What is the speed of the body far away from the earth (infinity)? Ignore the presence of sun and other planets. [NCERT]**

$v =$ velocity of projection; $v_e =$ escape velocity; $v = 2v_e$

$m =$ mass of the projectile; $M =$ mass of earth;

$R =$ radius of earth

At the surface of earth

$$K.E. = \frac{1}{2}mv^2; \quad P.E = -\frac{GMm}{R}$$

$$\text{Total energy} = K.E + P.E = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

$$\text{But, } \frac{1}{2}mv_e^2 = \frac{GMm}{R}$$

∴ Total energy of the body on the surface of earth

$$= \frac{1}{2}mv^2 - \frac{1}{2}mv_e^2 = \frac{1}{2}m(2v_e)^2 - \frac{1}{2}mv_e^2 = 3 \times \frac{1}{2}\mathrm{mv_e^2}$$

Let x be the velocity of the body at infinity.

$$\text{Total energy at infinity} = \frac{1}{2}K.E + P.E = \frac{1}{2}mx^2 + 0 = \frac{1}{2}mx^2$$

By law of conservation of energy

$$3 \times \frac{1}{2}mv_e^2 = \frac{1}{2}mx^2 \quad \therefore x = \sqrt{3} \times v_e = \sqrt{3} \times 11.2 = \mathbf{19.4\,kms^{-1}}$$

VII.27. Period of Saturn is 29.5 **years. Calculate the average distance of Saturn from the sun. (The radius of the earth's orbit is** 1.5×10^8 **km). [NCERT]**

$$(T_1/T_2)^2 = (d_1/d_2)^3$$

$T_1 = 1$ year, the period of the earth, $\quad d_1 = 1.5 \times 10^8$ km.

$T_2 = 29.5$ years; $\quad d_2 = ?$

$$\left(\frac{1}{29.5}\right) = \left(\frac{1.5 \times 10^8}{d_2}\right)^3$$

$$d_2^3 = (1.5 \times 10^8)^3 \times 29.5^2$$

$$d_2 = 1.5 \times 10^8 \times 29.5^{2/3} = \mathbf{14.3 \times 10^8\,km}$$

IMPORTANT POINTS

Kepler's laws of planetary motion

1. *The orbit of a planet round the sun lies in a plane containing the sun and it is an ellipse with sun at one of the foci.*
2. *Areal velocity is a constant*
3. $T^2 \alpha a^3$

$$\text{Binding energy of an orbiting satellite} = G\frac{Mm}{2r}$$

$$\text{Newton's law of gravitation } F = G\frac{m_1 m_2}{r^2}$$

$$\text{Acceleration due to gravity } g = \frac{GM}{R^2}$$

$$\text{Density of the earth } D = 3g/4\pi RG.$$

$$\text{Variation of } g \text{ with altitude } g' = g\left(1 - \frac{2h}{R}\right)$$

$$\text{Variation of } g \text{ with depth } g' = g(1 - h/R)$$

$$\text{Variation of } g \text{ with latitude } g' = g - R\omega^2 \cos^2 \lambda$$

$$\text{Orbital velocity of a satellite } V = \sqrt{\frac{GM}{r}}$$

$$\text{Minimum orbital velocity } V = \sqrt{\frac{GM}{R}} = \sqrt{gR}$$