SIMILPhysics Reference Text

CHAPTER III

MOTION IN A PLANE

VECTOR ALGEBRA

Physical quantities, in general, may be divided into two main classes. (1) *Scalar quantities* (2) *Vector quantities*.

Scalar quantities

These quantities have only magnitude and no direction. They are completely specified by a number and a unit. They obey the ordinary rules of algebra. Ex: Speed, distance, electric current, temperature, work etc.

Vector quantities

These quantities possess both magnitude and direction. They are added and subtracted according to special laws such as parallelogram law of addition, triangle law of addition etc. Ex: Force, velocity, acceleration, current density, intensity of electric field, angle, angular velocity etc.

Note: It is to be noted that all physical quantities having both magnitude and direction are not necessarily vectors. *All vectors obey the laws of vector algebra.* For example, the electric current and time have both magnitude and direction; but they are scalars, because they do not obey the laws of vector algebra.

Vectors associated with a linear directional effect are called *polar vectors*. Ex: Linear displacement, linear velocity, linear acceleration, force, linear momentum etc. The vectors associated with rotation about an axis are called *axial vectors*. Ex: Angular velocity, angular acceleration, torque, angular momentum etc.

Representation of a vector

(a) In print In print, a vector quantity is represented by a bold letter such as **A** or **a** and in writing it is represented by $\vec{A}$.

(b) Graphical

A vector is represented by a straight line with an arrow head; the length of the line representing its magnitude and the arrow head indicating its direction.

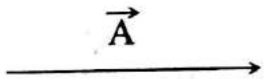


Fig. 5

Some important definitions about vectors

Equal vectors

Two vectors are said to be equal when they are identical both in magnitude and direction. Let $\vec{A}$ and $\vec{B}$ be two parallel vectors of equal magnitudes. Although they are situated at different places in space they are equal.

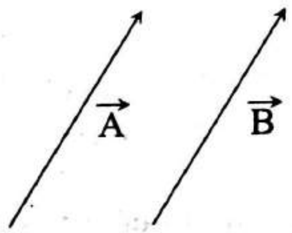


Fig. 6

Co-planar vectors

Vectors which are confined to the same plane are called co-planar vectors.

Position vector

A vector representing the position of a point with respect to an arbitrary origin is called position vector.

Let O be an arbitrary origin and P a point in space. The position vector of P with respect to O is represented by $\overrightarrow{OP}$

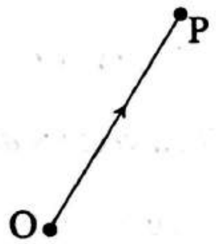


Fig. 7

Negative vector

A vector having the same magnitude but direction opposite to that of a given vector is called negative vector of the given vector (fig. 4).

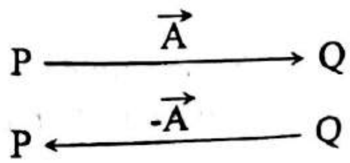


Fig. 8

Modulus of a vector

Modulus of a vector is the magnitude of the vector. The modulus of the vector $\vec{A}$ is represented as $|\vec{A}|$ or A.

Unit vector

It is a vector of unit magnitude drawn in the direction of a given vector. A unit vector in the direction of $\vec{A}$ is written as $\hat{A}$. It is used to specify a given direction. Thus we have

$$\vec{A} = |\vec{A}|\hat{A}$$

In cartesian coordinates $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along X, Y and Z axes respectively. They are called *orthogonal unit vectors*.

Zero vector

It is that vector which has zero magnitude and an arbitrary direction. A zero vector is represented by $\vec{0}$. It is also called a null vector.

The main properties of a zero vector

1. Zero or null vectors are taken to be equal and their directions are quite arbitrary and indeed quite immaterial.
2. The result of adding a zero vector to any vector is the vector itself. $\vec{A} + \vec{0} = \vec{A}$
3. The result of multiplication of a real number with zero vector is a zero vector itself and the result of multiplication of $\vec{0}$ and a vector $\vec{A}$ gives a zero vector.
   $n \times \vec{0} = \vec{0}$ and $\vec{0} \times \vec{A} = \vec{0}$
4. The result of addition of a vector to its own negative vector is a zero vector.
   $\vec{A} + (-\vec{A}) = \vec{0}$

Examples of zero vector

1. The velocity vector of a stationary object is a zero vector.
2. The acceleration of an object moving with uniform velocity is a zero vector.
3. The displacement of a stationary object is a zero vector.
4. The position vector of the origin of co-ordinate axes is a zero vector.

Addition of two vectors

1. When vectors are acting in the same direction

The magnitude of the resultant vector is equal to the sum of the magnitudes of the two vectors and the direction is the same as that of the two given vectors.

$$\vec{R} = \vec{A} + \vec{B};\ |\vec{R}| = |\vec{A}| + |\vec{B}|$$

2. When two vectors are acting in opposite directions

The magnitude of resultant vector is the difference in magnitude between the two vectors and the direction is that of the bigger vector.

$$\vec{R} = \vec{A} + (-\vec{B});\ |\vec{R}| = |\vec{A}| - |\vec{B}|$$

3. When two vectors are inclined at an angle

The sum of two vectors can be determined either by (i) the law of triangle of vectors or (ii) the law of parallelogram of vectors.

(i) Triangle law of vectors

If two sides of a triangle represent two given vectors in magnitude and direction and in the same order, then the third side of the triangle in the reverse order represents the vector sum of the vectors.

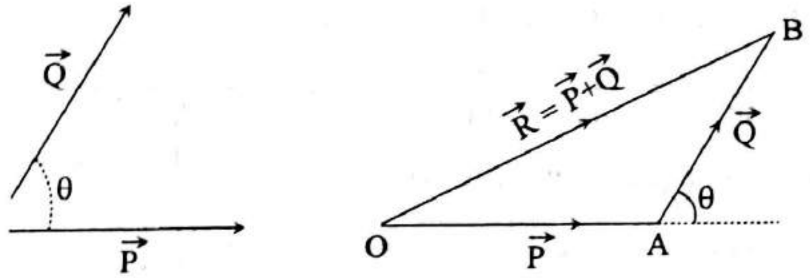


Fig. 9

If OA represents vector $\vec{P}$ and AB represents vector $\vec{Q}$ in the anticlockwise order, then $O\vec{B}$ in the clockwise order represents the vector sum $\vec{R}$ of $\vec{P}$ and $\vec{Q}$.

$$\vec{R} = \vec{P} + \vec{Q}$$

(ii) Parallelogram law of vectors

If two adjacent sides of a parallelogram represent two vectors in magnitude and direction, then the diagonal of the parallelogram starting from the point of intersection of the two vectors represents the vector sum of the two vectors.

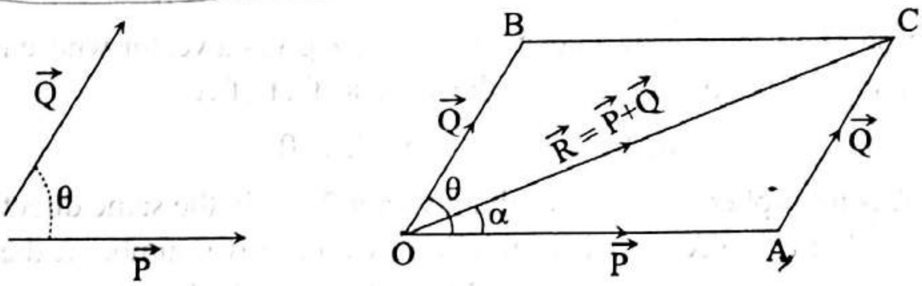


Fig. 10

If OA represents $\vec{P}$ and OB represents $\vec{Q}$, then the vector sum $\vec{R}$ is represented by the diagonal $\overrightarrow{OC}$ of the parallelogram $OACB$.

Analytical method of vector addition

Draw CD perpendicular to OA produced.

Let α be the angle which the resultant makes with the direction of $\vec{P}$.

In the triangle OCD (fig. 7), we have,

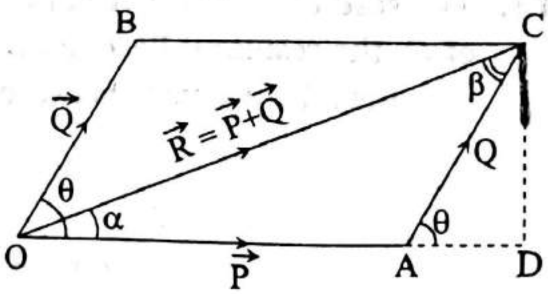


Fig. 11

$$OC^2 = OD^2 + CD^2 = (OA + AD)^2 + CD^2$$
$$= OA^2 + 2OA \times AD + AD^2 + CD^2$$
$$= OA^2 + AC^2 + 2OA \times AD$$

$$i.e., \; R^2 = P^2 + Q^2 + 2PQ\cos\theta$$

$$[\text{Since} \quad AD = AC\cos\theta]$$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

Direction of vector sum R

If α is the angle between the directions of the vector sum R and the vector P,

$$\tan\alpha = \frac{CD}{OD} = \frac{CD}{OA+AD} = \frac{Q\sin\theta}{P+Q\cos\theta}$$

Also from the triangle OAC (by the **Law of sines**)

$$\frac{R}{\sin\theta} = \frac{Q}{\sin\alpha} = \frac{P}{\sin\beta} \qquad \therefore \sin\alpha = \frac{Q}{R}\sin\theta$$

When two vectors are at right angles (fig. 8)

Here, $\theta = 90$. Then, $R = \sqrt{P^2+Q^2}$ and $\tan\alpha = Q/P$

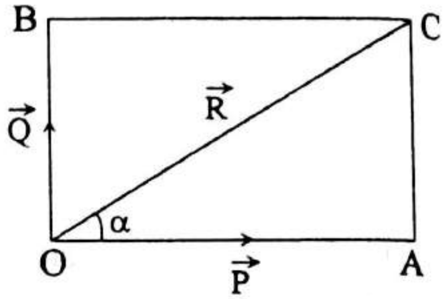


Fig. 12

Laws of vector addition

1. Vector addition is commutative: $\vec{A}+\vec{B} = \vec{B}+\vec{A}$
2. Vector addition is associative: $\vec{A}+(\vec{B}+\vec{C}) = \vec{B}+(\vec{C}+\vec{A}) = \vec{C}+(\vec{A}+\vec{B})$
3. Vector addition is distributive: $m(\vec{A}+\vec{B}) = m\vec{A}+m\vec{B}$

$$(m+n)\vec{A} = m\vec{A}+n\vec{A}$$

Multiplication of a vector by real numbers

Multiplication of a vector $\vec{A}$ by a positive number λ gives a vector whose magnitude is changed by the factor λ; but the direction is the same as that of $\vec{A}$

$$|\lambda\vec{A}| = \lambda|\vec{A}| \qquad \text{if} \quad \lambda > 0$$

For example if $\vec{A}$ is multiplied by 2 the resultant vector $2\vec{A}$ is in the same direction as $\vec{A}$ and has magnitude $|2\vec{A}|$. If the vector $\vec{A}$ is multiplied by a negative number λ the direction of the product is opposite to the direction of $\vec{A}$ but magnitude is $|\lambda\vec{A}|$

Examples

III.1. The resultant of two vectors P and Q is equal to R. On reversing the direction of Q, the resultant becomes S. Prove that $R^2+S^2 = 2(P^2+Q^2)$

Let θ be the angle between the directions of P and Q.

$$R^2 = P^2+Q^2+2PQ\cos\theta$$

$$S^2 = P^2+Q^2-2PQ\cos\theta$$

Adding $R^2+S^2 = 2(P^2+Q^2)$

III.2. $\hat{i}$ and $\hat{j}$ are unit vectors along the X and Y axes respectively. What is the magnitude and direction of the vector $\hat{i}+\hat{j}$?

$$|\hat{i}+\hat{j}| = \sqrt{1^2+1^2} = \sqrt{2}$$

If θ is the angle which $(\hat{i}+\hat{j})$ makes with X-axis,
then, $\tan\theta = 1/1 = 1 \therefore \theta = 45°$ with the direction of X-axis.

III.3. Establish the following vector inequalities. (i) $|\vec{a}+\vec{b}| \leq |\vec{a}|+|\vec{b}|$ and (ii) $|\vec{a}+\vec{b}| \geq |\vec{a}|-|\vec{b}|$

(i) $|\vec{a}+\vec{b}| = \sqrt{a^2+b^2+2ab\cos\theta}$; $|\vec{a}+\vec{b}|$ is maximum when $\cos\theta = 1$

$$\therefore |\vec{a}+\vec{b}| = \sqrt{a^2+b^2+2ab} = a+b = |\vec{a}|+|\vec{b}|$$

$$\therefore |\vec{a}+\vec{b}| \leq |\vec{a}|+|\vec{b}|$$

(ii) $|\vec{a}+\vec{b}| = \sqrt{a^2+b^2+2ab\cos\theta}$; $|\vec{a}+\vec{b}|$ is minimum when, $\cos\theta = -1$. Then,

$$|\vec{a}+\vec{b}| = \sqrt{a^2+b^2+-2ab} = a-b = |\vec{a}|-|\vec{b}|$$

$$\therefore |\vec{a}+\vec{b}| \geq |\vec{a}|-|\vec{b}|$$

III.4. The resultant of two vectors acting at an angle of 120° is perpendicular to the smaller vector. If the larger vector is of magnitude 10 units, find the resultant and the smaller vector.

Let R be the resultant and P the smaller vector. Then,

$$\text{Cos } 30 = \frac{R}{10} \therefore R = 10\cos 30 = \textbf{8.66 units}$$

$$\text{Sin } 30 = \frac{P}{10} \therefore P = 10\ \sin 30 = \textbf{5 units}$$

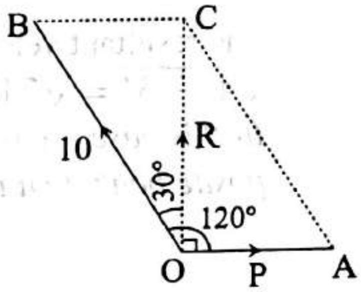


Fig. 13

III.5. On a certain rainy day, rain was falling vertically with a speed of $30\,\text{ms}^{-1}$. A wind started blowing after some time with a speed of $10\,\text{ms}^{-1}$ in the east to west direction. In which direction should a boy waiting at a bus stop hold his umbrella? [NCERT]

The resultant velocity of the rain makes an angle α with the vertical in the vertical plane containing the east-west direction such that,

$$\tan\alpha = \frac{10}{30} = 0.3333 \therefore \quad \alpha = 18°21'$$

The boy should hold his umbrella in the vertical plane making an angle of about 18° with the vertical towards east.

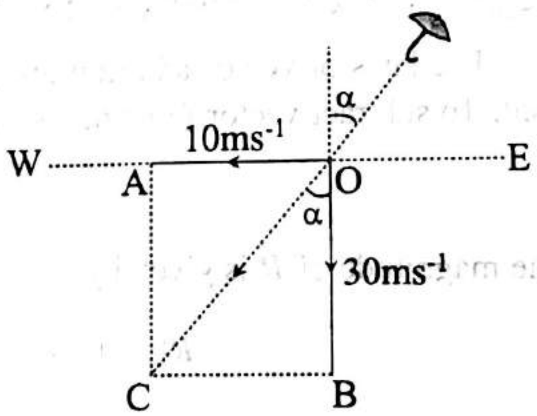


Fig. 14

III.6. *A boat moving at a speed of 4 km/h is required to cross a stream flowing at a speed of 3 km/h. (a) If the boat is pointed directly to the opposite bank, what will be its speed and direction while crossing the stream? If the width of the stream is 0.5 km, how far downstream will the boat reach the opposite bank? (b) If the boat is to reach directly to the opposite bank in which direction should the boat be pointed?

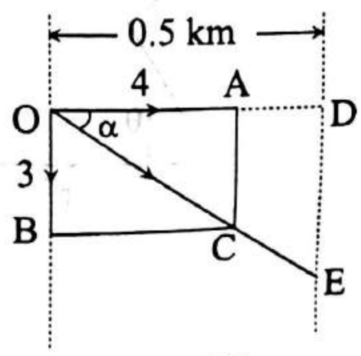


Fig. 15

Sides OA and OB of the parallelogram $OACB$ represent the velocity of the boat and the stream. The resultant velocity (speed) of the boat, $v = \sqrt{4^2 + 3^2} =$ **5 km/h** is along OC, the diagonal of the parallelogram. If v makes an angle α with OA, then, $\tan\alpha = 3/4 \therefore \alpha = 36°54'$. Hence the boat moves downstream at angle, $90 - 36°54' = 53°6'$ with the bank of the river.
The boat reaches the opposite bank at the point E downstream at a distance DE. From the figure, $\tan\alpha = DE/OD$;

$$\therefore DE = OD \times \tan\alpha = 0.5 \times (3/4) = \mathbf{0.375\,km}$$

(b) The resultant velocity of the boat and the current must be along OC'. If α is the angle which the direction of the boat makes with the normal to the bank.
The boat should be pointed along OA' upstream making an angle α with OD in order to reach directly the opposite bank at D. From the figure (12),

$$\text{Sin}\,\alpha = 3/4 \therefore \alpha = \mathbf{48°36'}$$

The resultant velocity in this case is
$\sqrt{4^2 - 3^2} = \sqrt{7}$ km/hr
In this case the boat takes a longer time to reach the opposite bank than in the first case.

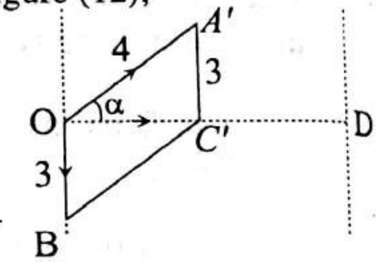


Fig. 16

Subtraction of two vectors

Subtraction of a vector $\vec{Q}$ from vector $\vec{P}$ is defined as the addition of vector $-\vec{Q}$ (negative of vector $\vec{Q}$) to the vector $\vec{P}$.

The laws of vector addition are applicable to the process of subtraction of vectors as well. To subtract vector $\vec{Q}$ from vector $\vec{P}$, we add to vector $\vec{P}$ the negative of vector $\vec{Q}$

$$\vec{P} - \vec{Q} = \vec{P} + (-\vec{Q})$$

The magnitude of $\vec{R}$ is given by

$$|\vec{R}| = |\vec{P} - \vec{Q}| = \sqrt{P^2 + Q^2 - 2PQ\cos\theta}$$

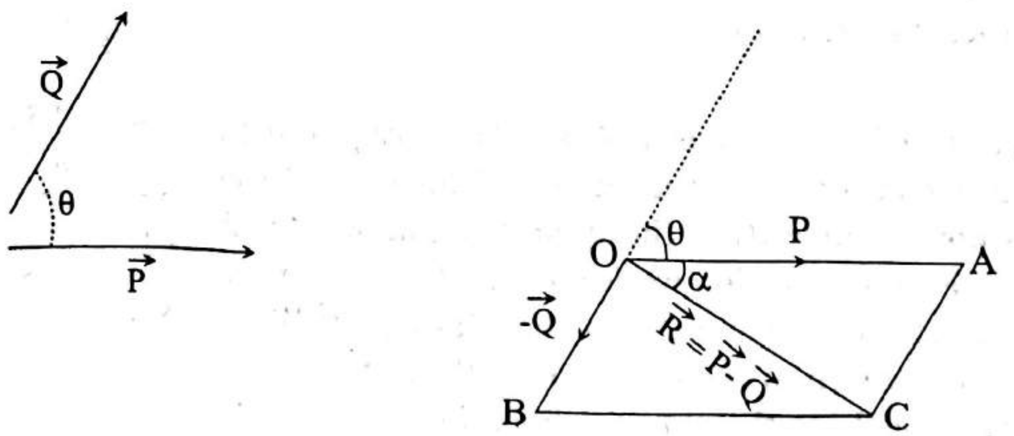


Fig. 17

$$\tan\alpha = \frac{Q\sin\theta}{P - Q\cos\theta}.$$

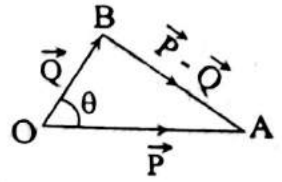


Fig. 18

Triangle method

If OA represents $\vec{P}$ and OB represents $\vec{Q}$, then $(\vec{P}-\vec{Q})$ is represented by the side $\overrightarrow{BA}$

RELATIVE VELOCITY

We consider a body to be at rest when its position does not change with respect to fixed objects in its surrounding. Usually we refer to the state of rest or motion with respect to the earth. But earth is moving with respect to the sun. Thus we cannot say a body to be at absolute rest or in a state of absolute motion.

Consider two trains moving on parallel tracks in the same direction with same speed. To an observer standing on the ground both the trains are moving with reasonable speed. But to an observer in one train, the other train appears to be not moving at all. The *relative velocity* of the train with respect to an observer in the other train is zero.

Consider two bodies A and B moving along a straight line with constant velocities V_A and V_B in the same direction. Then the velocity of A with respect to B is equal to $V_A - V_B$. If B is moving in the opposite direction, the relative velocity of A with respect to B is equal to $V_A + V_B$.

The relative velocity of B with respect to A is the velocity with which B appears to move when A is imagined to be at rest.

If $\vec{V}_A$ and $\vec{V}_B$ are the velocities of the bodies A and B, then the relative velocity of A with respect to B,

$$\vec{V}_{AB} = \vec{V}_A - \vec{V}_B.$$

The relative velocity of B with respect to A, $\vec{V}_{BA} = \vec{V}_B - \vec{V}_A$

Relative velocity in one dimensional motion

Consider two objects A and B moving with uniform velocity along a straight line with velocities v_A and v_B along parallel straight tracks in the same direction. Let $x_A(0)$ and $x_B(0)$ be the position co-ordinates of the particles at $t = 0$. At $t = t$, let the position coordinates be $x_A(t)$ and $x_B(t)$ respectively.

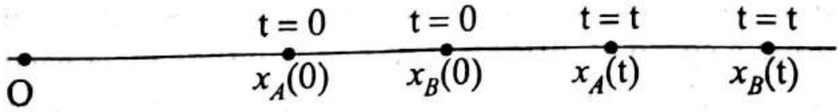


Fig. 19

Consider the equation, $S = vt$; *i.e.*, $x(t) - x(0) = vt$.

$$\therefore x_A(t) - x_A(0) = v_A t \;\therefore x_A(t) = x_A(0) + v_A t \quad \text{(i)}$$

$$\text{Similarly, } x_B(t) = x_B(0) + v_B t \quad \text{(ii)}$$

$$\text{Eqns: } (ii) - (i),\; x_B(t) - x_A(t) = [x_B(0) - x_A(0)] + (v_B - v_A)t \quad \text{(iii)}$$

$(v_B - v_A)$ is v_{BA}, the relative velocity of B with respect to A. $x_B(0) - x_A(0)$ is the distance between A and B at the instant, $t = 0$. This is a constant. $x_B(t) - x_A(t)$ is the distance between A and B at the instant $t = t$. This changes with time.

Thus the distance between A and B changes at a uniform rate of $(v_B - v_A)$, the relative velocity of B with respect to A.

If A just overtakes B at an instant t, $x_B(t) - x_A(t) = 0$. Then,

$$x_B(0) - x_A(0) + (v_B - v_A)t = 0$$

Position - time graphs - one dimensional relative motion

(i) Two bodies moving with same velocities

Consider two bodies A and B moving in the same direction with same velocities; B ahead of A.

$$x_B(t) - x_A(t) = x_B(0) - x_A(0) + (v_B - v_A)t$$

Since $v_A = v_B$, $x_B(t) - x_A(t) = x_B(0) - x_A(0)$, is a constant. *Hence position-time graphs are parallel (fig. 16).*

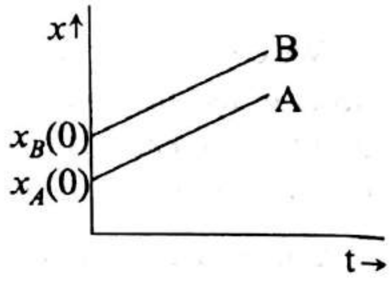


Fig. 20

(ii) Two bodies moving with different velocities

Here we can consider two cases.

Case (a): *The body B moving ahead of A with $v_B > v_A$*

$$x_B(t) - x_A(t) = x_B(0) - x_A(0) + (v_B - v_A)t.$$

Since $v_B > v_A$, $x_B(t) - x_A(t)$ increases with time. The graphs are straight lines with increasing separation between them with time (fig. 17).

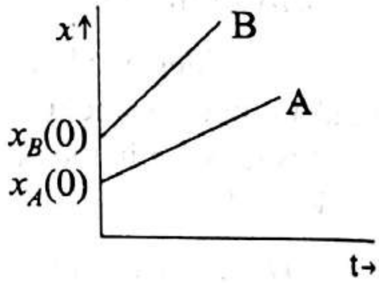


Fig. 21

Case (b): *The body B moving ahead of A with $v_A > v_B$*

Since $v_A > v_B$, $v_B - v_A$ is negative. So the distance between A and B decreases with time, becomes zero when they meet; and then A overtakes B and moves ahead of B (fig. 18).

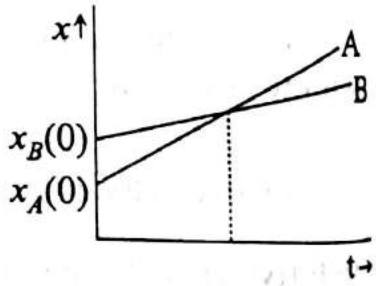


Fig. 22

Examples

III.7. Kochi is at a distance of 200 km from Trivandrum. A bus A sets out from Trivandrum at a speed of 30 km/h and another bus B sets out at the same time from Kochi at a speed of $20\,\text{km h}^{-1}$. When will they meet each other?

Let us take Trivandrum as the origin and Trivandrum to Kochi as positive direction.

$$x_A(0) = 0;\ x_B(0) = 200\text{km};\ \ v_A = 30\text{km/h};\ \ v_B = -20\text{km/h};$$

Let the buses meet at the instant t; *i.e.*, at the instant t, $x_B(t) - x_A(t) = 0$. But,

$$x_B(t) - x_A(t) = x_B(0) - x_A(0) + (v_B - v_A)t$$

$$0 = (200 - 0) + (-50 \times t)\ \therefore t = \mathbf{4\,hr}$$

The buses A and B meet after 4 h from the start.

III.8. A police van moving on a highway with a speed of $30\,kmh^{-1}$ fires a bullet at a thief's car speeding away in the same direction with a speed of $192\,kmh^{-1}$. If the muzzle speed of the bullet is $150\,ms^{-1}$, with what speed does the bullet hit the thief's car? [NCERT]

$$\text{Muzzle velocity of the bullet} = 150\ ms^{-1} = 540\ kmh^{-1}$$

$$\text{Effective velocity of bullet} = 540 + 30 = 570\ kmh^{-1}$$

$$\text{Speed of the bullet w.r.t the theif's car} = 570 - 192 = \mathbf{378\ kmh^{-1}}$$

III.9. A jet aeroplane travelling at the speed of 500 km/h ejects the product of combustion at the speed of 1200 km/h relative to the plane. What is the speed of the latter with respect to an observer on ground? [NCERT]

Let v_g be the velocity of the gas and v_p that of the plane. $v_p = 500$ km/h; $v_{gp} = v_g - v_p = -1200$ km/h (it is negative since it moves opposite to the direction of motion of the plane).

$$v_g - 500 = -1200; \quad v_g = -1200 + 500 = \mathbf{-700 km/h}$$

III.10. On a two lane road, car A is travelling with a speed of 36 km/h. Two cars B and C approach the car A in opposite directions with speed of 54 km/h each. At a certain instant the distance $AB = AC = 1$ km, B decides to overtake A before C does. What minimum acceleration of car B is required to avoid an accident? [NCERT]

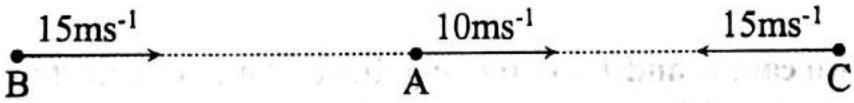


Fig. 23

$$\text{Relative velocity of } B \text{ with respect to } A = 15 - 10 = 5 = 5\ ms^{-1}$$

$$\text{Relative velocity of } C \text{ with respect to } A = 15 + 10 = 25\ ms^{-1}$$

$$\text{Time taken by } C \text{ to cross } A = \frac{1000}{25} = \mathbf{40\ s.}$$

To avoid an accident B should cover a distance of 1000 m in less than 40 s. Hence B should be accelerated.

Consider the motion of B with respect to A.

$$S = 1000\ m;\ u = 5ms^{-1};\ t = 40\ s;\ a = ?$$

$$S = ut + (1/2)at^2;\ 1000 = 5 \times 40 + (1/2) \times a \times 40^2$$

$$1000 = 200 + 800a;\ 800a = 800\ \therefore a = \mathbf{1\ ms^{-2}}$$

The minimum acceleration of car $\mathbf{B = 1\ m\,s^{-2}}$

Relative velocity: Two dimensional motion

Consider two objects A and B moving in different directions. Let v_A and v_B be the velocities of the two objects and θ the angle between the two directions. The relative velocity of A with respect to B is the vector difference of the velocities of A and B

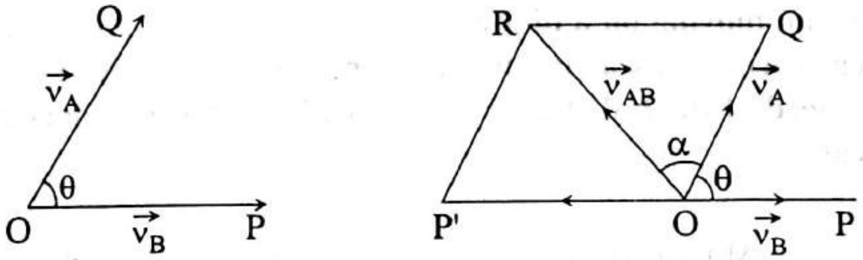


Fig. 24

Relative velocity of A with respect to B,

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

This is obtained by subtracting the velocity of B from velocity of A. In magnitude, the relative velocity of A with respect to B,

$$v_{AB} = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$$

If α is the angle which the direction of $\vec{V_{AB}}$ makes with the direction of A, then,

$$\tan\alpha = \frac{v_B \sin\theta}{v_A - v_B\cos\theta}.$$

Also

$$\frac{v_{AB}}{\sin\theta} = \frac{v_B}{\sin\alpha}; \qquad \sin\alpha = \frac{v_B \sin\theta}{v_{AB}}$$

Examples

III.11. Two cars A and B are moving due east and due north directions with velocities 8 ms^{-1} and 6 ms^{-1} respectively. Calculate the relative velocity of B with respect to A.

Relative velocity of B with respect to A

$$\vec{V}_{BA} = \vec{V}_B - \vec{V}_A$$

$$V_{BA} = \sqrt{8^2 + 6^2} = \mathbf{10\ ms^{-1}},$$

$$\text{Tan}\,\alpha = \frac{8}{6} = 1.33 \quad \therefore \alpha = \mathbf{53°}$$

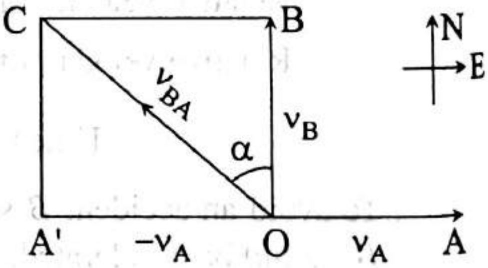


Fig. 25

III.12. To a person going due east in a car with velocity 25 km/h, a train appears to move due north with velocity $25\sqrt{3}$ km/h. What is the actual velocity and direction of the train?

Relative velocity of the train w.r.t the car = velocity of the train–velocity of the car.
Relative velocity of the train $25\sqrt{3}$ km/hr must be the resultant of the actual velocity of the train and reverse velocity of the car.
Let v be the velocity of the train and θ the angle which the actual direction of the train makes with the direction of the car.

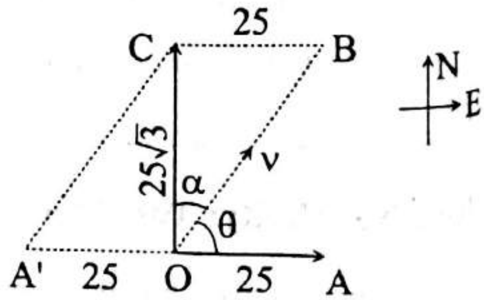


Fig. 26

$$\text{Tan}\,\alpha = \frac{25}{25\sqrt{3}} = \frac{1}{\sqrt{3}}; \quad \alpha = 30^\circ \quad \therefore \theta = \mathbf{60^\circ}$$

$$v = \sqrt{(25\sqrt{3})^2 + 25^2} = 25\sqrt{3+1} = \mathbf{50\ km/h}$$

The velocity of the train is **50 km/h** in a direction **60°** north of east.

Resolution of a vector into two components along given directions

Let the vector $\vec{R}$ is to be resolved into two components in the directions of $\vec{A}$ and $\vec{B}$

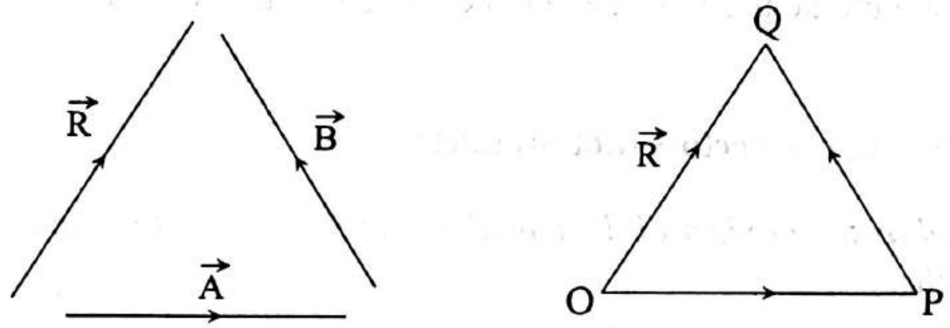


Fig. 27

Draw OQ to represent $\vec{R}$ in magnitude and direction. From the point O draw a line parallel to the vector $\vec{A}$ and from Q draw a line parallel to $\vec{B}$. The two lines intersect at P. From triangle law of vector addition,

$$\vec{OQ} = \vec{OP} + \vec{PQ}$$

Therefore $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ are two component vectors of $\vec{R}$ in the directions of $\vec{A}$ and $\vec{B}$.

Let $\overrightarrow{OP} = \lambda\vec{A}$ and $\overrightarrow{PQ} = \mu\vec{B}$; where λ and μ are real numbers

Therefore, $\vec{R} = \lambda\vec{A} + \mu\vec{B}$

Rectangular components of a vector in a plane

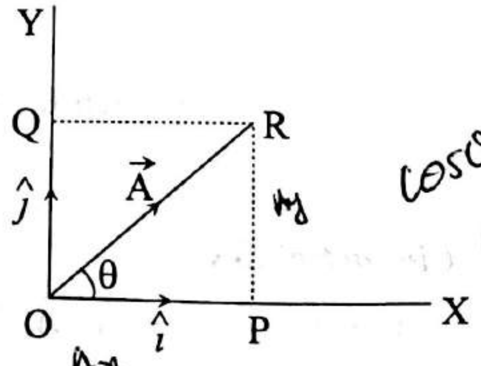


Fig. 28

Consider a vector $\vec{A}$ which is to be resolved in the directions of unit vectors $\hat{i}$ and $\hat{j}$ along the X, Y axes.

Let OR represents $\vec{A}$. From R, drop perpendiculars RP and RQ to OX and OY respectively. From parallelogram law of addition of vectors,

$$\vec{OR} = \vec{OP} + \vec{OQ}; \quad \text{Then } \vec{A} = \vec{A}_x + \vec{A}_y;$$

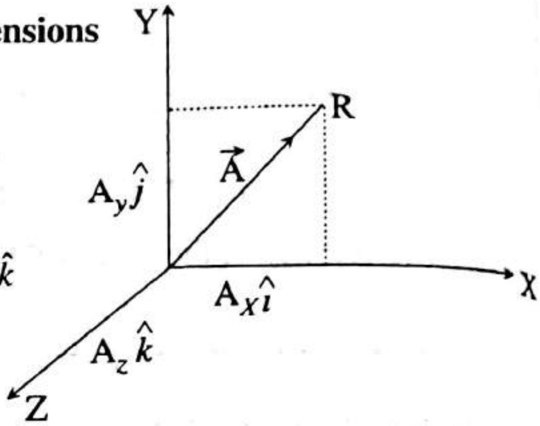
where A_x and A_y are the components of A along X and Y axes.

$$\vec{A} = A_x\hat{i} + A_y\hat{j}; \quad \text{Also} \quad A_x = A\cos\theta; \text{ and } A_y = A\sin\theta$$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \tan\theta = A_y/A_x$$

Rectangular components of a vector in three dimensions

Let A_x, A_y, A_z be the rectangular components of a vector in the X and Y and Z directions. Then,



$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z; \quad \vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$|\vec{A}| = \sqrt{\mathbf{A_x^2} + \mathbf{A_y^2} + \mathbf{A_z^2}}$$

Fig. 29

Multiplication of vectors

The product of two vectors is not necessarily a vector quantity. It may be a scalar or a vector depending on the nature of the product. Accordingly there are two types of vector products.

(i) Scalar product of two vectors (dot product)

It is defined as the product of the magnitude of the two vectors and the cosine of the angle between them

If $\vec{A}$ and $\vec{B}$ are two vectors and θ the angle between them, the scalar product is given by,

$$\vec{A} \cdot \vec{B} = AB\cos\theta; \text{ read as } A \text{ dot } B \text{ equals } AB\cos\theta.$$

The scalar product can be expressed in two ways.

$$\vec{A} \cdot \vec{B} = A(B\cos\theta) = \text{ magnitude of } \vec{A} \times \text{ resolved component of } \vec{B} \text{ along } \vec{A}$$
$$= B(A\cos\theta) = \text{ magnitude of } \vec{B} \times \text{ resolved component of } \vec{A} \text{ along } \vec{B}$$

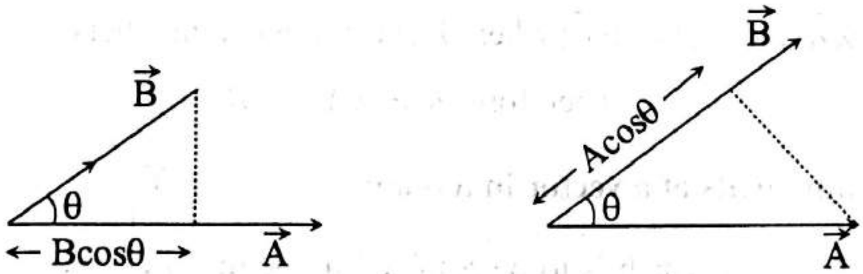


Fig. 30

Characteristics

1. Since $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$, the scalar product is commutative
2. If $\vec{A} \cdot \vec{B} = 0$, then the two vectors are at right angles
3. If two vectors are parallel $\vec{A} \cdot \vec{B} = AB$
4. $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1;$ since $\theta = 0$ and
   $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0;$ since $\theta = 90°$
5. In Cartesian co-ordinates

$$\vec{A} \cdot \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \cdot (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$
$$= A_xB_x + A_yB_y + A_zB_z$$

(ii) Vector product (Cross product) of two vectors

The vector product of two vectors $\vec{A}$ and $\vec{B}$ is written as $\vec{A} \times \vec{B}$ (read as A cross B) and is a vector whose magnitude is equal to the product of the magnitudes of the given vectors and the sine of the angle between them.

$$\vec{A} \times \vec{B} = AB \sin\theta$$

The direction of the product is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ and is given by the right hand screw rule.

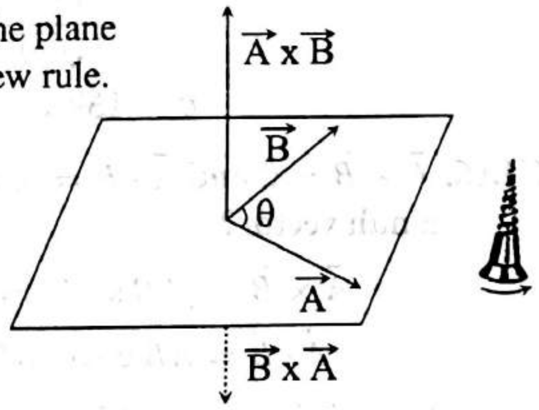


Fig. 31

Right hand screw rule

Imagine the rotation of a right handed screw whose axis is perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$. If we turn the screw from $\vec{A}$ to $\vec{B}$ then the direction of advancement of the screw is the direction of $\vec{A} \times \vec{B}$.

Characteristics of vector product

1. The vector product of two vectors does not obey commutative law.
   $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$. But, $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$
2. The vector product of two collinear vectors is zero. $\vec{A} \times \vec{B} = \vec{0}$, if $\theta = 0$ or π
3. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$; Also $\hat{i} \times \hat{j} = \hat{k}$; $\hat{j} \times \hat{k} = \hat{i}$; $\hat{k} \times \hat{i} = \hat{j}$
4. The vector product of two perpendicular vectors is given by $\vec{A} \times \vec{B} = AB\hat{n}, \theta = 90°$
5. In Cartesian coordinates $\vec{A} \times \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$
   In determinant form,
   $$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = i(A_yB_z - A_zB_y) - j(A_xB_z - A_zB_x) + k(A_xB_y - A_yB_x)$$

Examples

III.13. Show that the scalar product of two vectors is equal to the sum of the products of their corresponding x, y, z components.

Let $A = A_xi + A_yj + A_zk$ and $B = B_xi + B_yj + B_zk$ be the two vectors.

Their scalar product,

$$A \cdot B = (A_xi + A_yj + A_zk).(B_xi + B_yj + B_zk)$$
$$= \mathbf{A_xB_x + A_yB_y + A_zB_z}$$

(Since $i \cdot i = j \cdot j = k \cdot k = 1$ and $i \cdot j = j \cdot k = k \cdot i = 0$)

III.14. The force F and displacement d of a particle are given by $\vec{F} = (2i + 3j + 5k)N$ and $\vec{d} = (i + 2j + 3k)m$. Evaluate the work done.

Work done, $\boldsymbol{W = \vec{F} \cdot \vec{d}} = (2i + 3j + 5k) \cdot (i + 2j + 3k)$

$= 2 + 6 + 15 =$ **23 joules**

III.15. What is the angle between the vectors, $\vec{A} = 2i - 3j + 4k$ and $\vec{B} = -2i + 5j - 3k$?

$$\text{Since } A \cdot B = AB\cos\theta \;\therefore\quad \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|AB|}$$

$$\text{Cos}\,\theta = \frac{(2i - 3j + 4k)\cdot(-2i + 5j - 3k)}{\sqrt{2^2 + (-3)^2 + 4^2} \times \sqrt{(-2)^2 + 5^2 + (-3)^2}}$$

$$= \frac{-4 - 15 - 12}{\sqrt{29 \times 38}} = \frac{-31}{\sqrt{29 \times 38}} = -0.9340$$

$$\theta = \mathbf{159°6'} \;\therefore\quad (\text{Since } \cos(180 - \theta) = -\cos\theta)$$

III.16. $\vec{A} \times \vec{B} = 0$ and $\vec{A}\cdot\vec{B} = 0$, does it imply that one of the vectors A or B must be a null vector?

$$\vec{A} \times \vec{B} = |AB\sin\theta| = 0 \quad i.e.,\ A = 0, \quad B = 0 \quad \text{or} \quad \sin\theta = 0$$

$$\vec{A}\cdot\vec{B} = AB\cos\theta = 0 \quad i.e.,\ A = 0, \quad B = 0, \text{ or} \quad \cos\theta = 0$$

Since it is not possible to have both $\sin\theta$ and $\cos\theta$ zero, it is obvious that either $\vec{A} = \vec{0}$ or $\vec{B} = \vec{0}$

III.17. Show that the vectors $\vec{A} = 2\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{B} = -6\hat{i} + 9\hat{j} + 3\hat{k}$ are parallel

If vectors A and B are parallel then $\vec{A} \times \vec{B} = AB\sin\theta = 0$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & -1 \\ -6 & +9 & +3 \end{vmatrix} = \hat{i}(-9 + 9) - \hat{j}(6 - 6) + \hat{k}(18 - 18) = 0$$

Therefore $\vec{A}$ and $\vec{B}$ are parallel.

III.18. Determine the unit vector perpendicular to both $\vec{A} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{B} = \hat{i} - \hat{j} + 2\hat{k}$

Let $\vec{C}$ be perpendicular to both $\vec{A}$ and $\vec{B}$, then $\vec{C}$ can be taken as the cross product of $\vec{A}$ and $\vec{B}$

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i}(2 + 1) - \hat{j}(4 - 1) + \hat{k}(-2 - 1)$$

$$= 3\hat{i} - 3\hat{j} - 3\hat{k}$$

$$|\vec{C}| = \sqrt{3^2 + (-3)^2 + (-3)^2} = \sqrt{27} = 3\sqrt{3}$$

$$\text{Unit vector along } \vec{C} = \hat{C} = \frac{\vec{C}}{|\vec{C}|} = \frac{3\hat{i} - 3\hat{j} - 3\hat{k}}{3\sqrt{3}} = \frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$$

III.19. Consider two vectors $\vec{F} = (4\hat{i} - 10\hat{j})$ newton and $\vec{r} = (-5\hat{i} - 3\hat{j})\ m$. Compute $(\vec{r} \times \vec{F})$ and state what physical quantity does it represent?

$$\vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & -3 & 0 \\ 4 & -10 & 0 \end{vmatrix} = \hat{i}(0 - 0) - \hat{j}(0 - 0) + \hat{k}(50 + 12) = 62\,\hat{k}$$

The physical quantity represents a torque.

PROJECTILES

Projectile *is the name given to a body which after having been given an initial velocity is allowed to move under the influence of gravity alone.*

Body projected horizontally from a height

Consider a body projected horizontally from a height h with uniform velocity u. When it is released it has uniform velocity in the forward direction and acceleration in the downward direction. The two motions are physically independent. Hence the time taken by the particle to reach the ground level is the same as time for free fall.

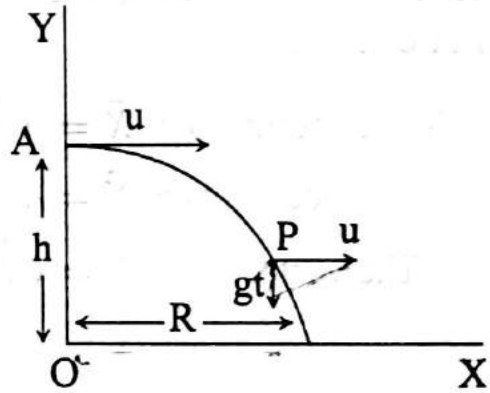


Fig. 34

i.e., For the vertical displacement, $u = 0;\ a = g;\ S = h$

$$\therefore \quad \text{Time for free fall, } t = \sqrt{2h/g}$$

During this time the particle has been displaced horizontally with constant velocity u. Hence,

$$\text{Range, } R = u \times t = u\sqrt{2h/g}$$

If P is the position of the particle t seconds after projection,

$$\text{velocity at P} = \sqrt{u^2 + (gt)^2}$$

Body projected at an angle with the horizontal

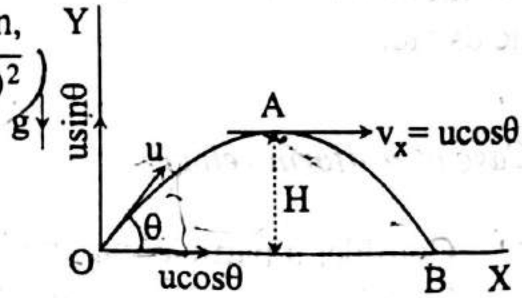


Fig. 35

Consider a particle projected with a velocity u in a direction making an angle θ with the horizontal. The velocity can be resolved in horizontal and vertical directions.

$$\text{Horizontal component,} \quad u_x = u\cos\theta$$

$$\text{Vertical component,} \quad u_y = u\sin\theta$$

$$\text{Horizontal displacement in time } t, \quad x_t = u_x t$$

$$\text{Vertical displacement in time } t, \quad y_t = u_y t - (1/2)gt^2$$

The horizontal component remains constant throughout the motion. The vertical component alone is affected by gravity.

Variation of vertical and horizontal components of velocities

A projectile motion is a two dimensional motion with acceleration, $a = g$, only in one direction, *i.e.*, vertically downwards. Since g has no component in the horizontal direction, horizontal component of velocity $u_x = u\cos\theta$ remains uniform throughout the fight.

Since g acts vertically down, vertical component of velocity $u_y = u\sin\theta$ continuously changes. As the projectile moves form O to A (Fig. 31), the vertical component of velocity decreases from $u\sin\theta$ to 0. At the maximum height, *i.e.*, at the point A, the vertical component of velocity becomes zero. As the projectile moves along AB, v_y increases from 0 to $u\sin\theta$.

Since v_y is zero at the maximum height, resultant velocity at the maximum height is $v_x = u\cos\theta$

At any instant t, $v_y = u_y - gt = u\sin\theta - gt$. Resultant velocity at any instant t, $v = \sqrt{v_x^2 + v_y^2}$. If it makes an angle α with the horizontal $\tan\alpha = v_y/v_x$.

Expressions for maximum height, time of flight and range of a projectile

(1) Maximum height reached (H)

At the maximum height, vertical component of velocity is zero.

Initial vertical velocity, $u_y = u\sin\theta$

Acceleration, $a = -g$

Final vertical velocity, $v_y = 0$

Vertical displacement, $S = H$

$$v^2 = u^2 + 2aS; \quad 0 = u^2\sin^2\theta - 2gH$$

$$\mathbf{H} = \frac{u^2\sin^2\theta}{2g}$$

It may be noted that at maximum height, the particle has a horizontal velocity, $v_x = u\cos\theta$.

Time of flight (T)

It is the time taken by the projectile to return to the same horizontal level as the point of projection.

Initial vertical velocity, $u_y = u\sin\theta$

Vertical displacement $= 0$

Acceleration, $a = -g$

Time of flight, $t = T$

$$S = ut + (1/2)at^2; \quad 0 = u\sin\theta \times T - (1/2)gT^2 \qquad \therefore T = \frac{2u\sin\theta}{g}$$

Horizontal Range (R)

It is the horizontal displacement of the particle through the point of projection. Throughout the time of flight the particle was moving horizontally with constant velocity $u\cos\theta$.

$$\text{Range, } R = u\cos\theta \times T = \frac{u\cos\theta \times 2u\sin\theta}{g} = \frac{2u^2\sin\theta\cos\theta}{g};$$

$$\therefore \mathbf{R = \frac{u^2 \sin 2\theta}{g}}.$$

Maximum Range (R_{max}) for a given velocity

For a given velocity of projection the range is maximum when $\sin 2\theta = 1$; $2\theta = 90°$ or $\theta = 45°$ $\therefore \mathbf{R_{max} = u^2/g}$.

Two angles of projection for the same horizontal range

For a given velocity of projection, in general, there are two angles of projection to obtain the same range and these angles are equally inclined to the direction of maximum range.

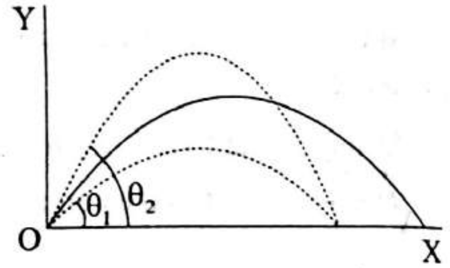


Fig. 36

Let θ_1 and θ_2 be the two angles of projection to obtain the same range

$$R = \frac{u^2\sin 2\theta_1}{g} = \frac{u^2\sin 2\theta_2}{g}$$

$$\therefore \sin 2\theta_1 = \sin 2\theta_2; \quad 2\theta_1 + 2\theta_2 = 180° \quad \therefore \theta_1 + \theta_2 = 90°$$

Thus θ and $(90 - \theta)$ are the two angles of projection to obtain the same range. These directions are equally inclined to the angle for maximum range, *i.e.*, 45°.

**To show that the path of a projectile is a parabola*

Let u be initial velocity of projection and θ angle of projection.

Initial vertical component $= u\sin\theta$

Horizontal component $= u\cos\theta$

After t seconds, the vertical displacement,

$$y = u\sin\theta \times t - (1/2)gt^2 \tag{1}$$

The horizontal displacement,

$$x = u\cos\theta \times t \;\therefore\; t = \frac{x}{u\cos\theta} \tag{2}$$

$$\therefore y = u\sin\theta \times \frac{x}{u\cos\theta} - \frac{1}{2}g \times \frac{x^2}{u^2\cos^2\theta}$$

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

This is of the form $y = bx - cx^2$ and is the equation of a parabola. Hence the path of a projectile is a parabola.

Examples

III.20. A particle is projected with a velocity 49 ms^{-1} at an angle of $30°$ to the horizontal. Calculate the maximum height, time of flight and the horizontal range.

$$u = 49 \text{ ms}^{-1}; \quad \theta = 30°; \; H = ?; \; T = ?; \; R = ?$$

$$\text{Maximum height, } H = \frac{u^2 \sin^2 \theta}{2g} = \frac{49^2 \sin^2 30}{2 \times 9.8} = \mathbf{30.625 \text{ m}}$$

$$\text{Time of flight, } T = \frac{2u \sin \theta}{g} = \frac{2 \times 49 \times \sin 30}{9.8} = \mathbf{5 \text{ s}}$$

$$\text{Horizontal range, } R = \frac{u^2 \sin 2\theta}{g} = \frac{49^2 \sin 60}{9.8} = \mathbf{212.17 \text{ m}}$$

III.21. A shot is fired from a gun on the top of a hill 90 m high with a velocity 80 ms^{-1} at an angle $30°$ with the horizontal. Find the horizontal distance between the vertical line through the gun and the point where the shot strikes the ground. $g = 9.8 \text{ ms}^{-2}$.

At $t = 0$, let O be the position of the shot. The horizontal and vertical components of velocities are $u \cos \theta$ and $u \sin \theta$. Let t be the time taken by the shot to reach the ground.

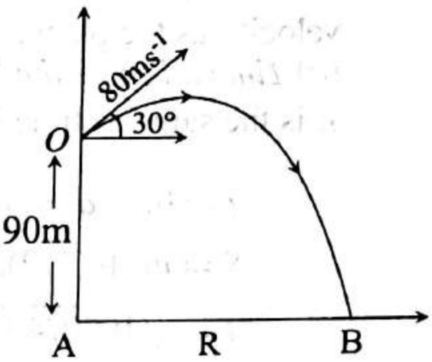


Fig. 37

To find the time taken by the shot to reach the ground:

Consider the vertical displacement, $S = -h = -90$ m.

$$a = -g = -9.8\text{ms}^{-2}; \; u_y = u \sin \theta = 80 \sin 30;$$

Consider the equation, $S = ut + (1/2)gt^2$.

$$i.e., \; -90 = 80 \sin 30 \times t - 4.9t^2; \quad -90 = 40t - 4.9t^2$$

$$49t^2 - 400t - 900 = 0; \quad t = \mathbf{10 \text{ s}}$$

To find the horizontal displacement:

During this time, the shot moves horizontally with a uniform velocity.

$$\therefore \text{Displacement} = \text{velocity} \times \text{time.}$$

$$\text{Horizontal displacement} = 80 \cos 30 \times 10 = \mathbf{692.8 \text{ m}}$$

III.22. The ceiling of a long hall is 25 m high. What is the maximum horizontal distance that a ball thrown with a speed of 40 ms^{-1} can go without hitting the ceiling of the hall? [NCERT]

$H = 25$ m; $u = 40$ ms^{-1}; $g = 9.8$ ms^{-2}; $\theta = ?$; $R = ?$

$$H = \frac{u^2 \sin^2\theta}{2g}; \quad 25 = \frac{40^2 \sin^2\theta}{2 \times 9.8};$$

$$\text{Sin}\,\theta = \sqrt{\frac{25 \times 2 \times 9.8}{40^2}} = 0.5534 \quad \therefore \theta = 33°36'$$

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{40^2 \sin 67°12'}{9.8} = \mathbf{150.5\ m}$$

III.23. A plane flying horizontally at $100\,\text{ms}^{-1}$ at a height of 1000 m releases a bomb from it. Find the (a) time taken by the bomb to reach the ground (b) the velocity with which it hits the target on the ground (c) distance of the target.

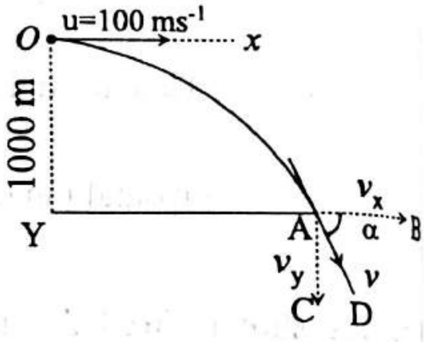


Fig. 38

At $t = 0$, let the bomb be at O just released from the plane. Then it has the same horizontal velocity as the plane.

(a) Time taken by the bomb to reach the ground:-
It is the same as time for free fall

$u = 0$; $a = +9.8$ ms^{-2}; $S = +1000$ m; $t = ?$

$S = ut + (1/2)at^2$; $1000 = 4.9t^2$

$t = \sqrt{1000/4.9} = \mathbf{14.28\ s}$

(b) To calculate the velocity of the bomb on hitting the target

Horizontal component of velocity at A, $v_x = 100$ ms^{-1}

(Since the horizontal velocity remains uniform throughout the flight)

Vertical component of velocity, $v_y = 0 + 9.8t = 9.8 \times 14.28$
$= 140$ ms^{-1}

Resultant velocity, $v = \sqrt{v_x^2 + v_y^2} = \sqrt{100^2 + 140^2} = \mathbf{172.1\ ms^{-1}}$

Let $\vec{v}$ be at angle α with the horizontal.

$$\text{Tan}\,\alpha = \frac{v_y}{v_x} = \frac{140}{100} = 1.4 \therefore \alpha = 54°28'$$

(c) Horizontal displacement (Horizontal distance of the target A from Y)
Since the horizontal component of the velocity 100 ms^{-1} remains uniform,

Horizontal displacement $= 100 \times 14.28 = \mathbf{1428\ m}$

III.24. Prove that the time of flight T and the horizontal range R of a projectile are connected by the equation $gT^2 = 2R\tan\theta$; where θ is the angle of projection.

Let u be the velocity of projection.

$$T = \frac{2u\sin\theta}{g} \text{ and } R = \frac{u^2 sin2\theta}{g} = \frac{2u^2\sin\theta\cos\theta}{g}$$

LHS; $gT^2 = g \times \frac{4u^2\sin^2\theta}{g^2} = \frac{4u^2\sin^2\theta}{g}$

RHS; $2R\tan\theta = 2 \times \frac{2u^2\sin\theta\cos\theta}{g} \times \frac{\sin\theta}{\cos\theta} = \frac{4u^2\sin^2\theta}{g}$

$\therefore$ $\mathbf{gT^2 = 2R\tan\theta}$

III.25. A bomb is released from an aeroplane when it was at a height of 1960 m above a point A on the ground and was moving horizontally at a speed of 100 ms^{-1}. Find the distance from A to the point where the bomb strikes the ground. $g =$ 9.8 ms^{-2}

At the point of dropping, the bomb has the same velocity as the aeroplane.

Horizontal velocity of the bomb at the time of release $= 100\ ms^{-1}$

Time taken by the bomb to reach the ground

$u = 0$; $a = +9.8\ ms^{-2}$; $S = 1960$ m; $t = ?$

$S = ut + (1/2)at^2$; $1960 = 4.9\,t^2$; $t^2 = 400$; $t = \mathbf{20\ s}$

Horizontal distance covered in 20 s $= 100 \times 20 =$ **2000 m**

III.26. A fighter plane flying horizontally at an altitude of 1.5 km with a speed of 720 km/h passes directly over an anti–aircraft gun. At what angle from the vertical should the gun be fired, for the shell with muzzle speed 600 ms^{-1}, to hit the plane? At what minimum altitude should the pilot fly the plane to avoid being hit?

[NCERT]

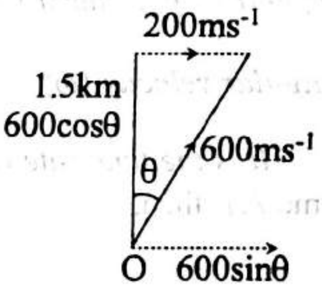


Fig. 39

Velocity of the plane $= 720$ km/h $= 200\ ms^{-1}$

Let the shell be fired at an angle θ with the vertical.

Horizontal component of the velocity of the shell $= 600\ \sin\theta$

Vertical component of the velocity of the shell $= 600\ \cos\theta$

For the shell to hit the plane, the horizontal distance covered by the shell and the plane must be same in the same time t

$$\therefore\ 600\sin\theta \times t = 200 \times t; \quad \text{Sin}\,\theta = \frac{200}{600} = 0.3333 \quad \therefore \theta = \mathbf{19°28'}$$

To avoid hitting, the plane should fly above the maximum height reached by the shell.

Maximum vertical height that can be reached by the shell.

Initial vertical velocity, $u = 600\cos 19°28'$; Final vertical velocity, $v = 0$;

$a = -9.8\ ms^{-2}$; $S = ?$

Consider the equation of motion, $v^2 - u^2 = 2aS$.

$$i.e., 0 - (600 \times \cos 19°28')^2 = -2 \times 9.8 \times S$$

$$\therefore S = 600^2 \cos^2 19°28'/2 \times 9.8$$

$$= 16320 \text{ m} = \mathbf{16.32\ km}$$

The plane should fly at an altitude greater than 16.32 km to avoid being hit.

CIRCULAR MOTION

We are familiar with objects moving in a circular path. Familiar examples are motion of the earth round the sun, motion of moon round the earth etc.

Angular displacement (θ)

The angle swept over by the radius vector in a given interval of time is called angular displacement. It is measured in radian and is a *vector quantity* provided θ is small.

The direction of the displacement is perpendicular to the plane of rotation and along the axis and is given by ***Right hand grip rule****. If the curvature of the fingers of the right hand represents the sense of rotation of the object, the thumb represents the direction of the angular displacement vector.* Angular displacement is an **axial vector**.

Angular velocity (ω)

It is the time rate of change of angular displacement. If $\Delta\theta$ is the angle described in time Δt, then,

$$\text{Angular velocity} \quad \omega = \frac{\Delta\theta}{\Delta t}$$

Angular velocity is measured in *radians/second* (rad.s^{-1}) and is a *vector quantity*. Its direction is same as that of $\Delta\theta$.

Instantaneous angular velocity

It is the angular velocity of the particle at an instant

$$\text{Instantaneous angular velocity, } \omega = Lt_{\Delta t \to 0}(\Delta\theta/\Delta t) = \frac{d\theta}{dt}$$

Uniform circular motion

When a point object moves round a circle with constant speed, the motion of the object is said to be uniform circular motion.

Time period (T)

It is the time taken by the particle to describe one complete revolution.

Frequency (ν)

It is defined as the number of revolutions made by the object in one second; $\nu = 1/T$

Relation connecting ω, T ***and*** ν

Angular velocity, $\omega = \theta/t = 2\pi/T = 2\pi\nu$

Relation between linear velocity and angular velocity

Consider a particle moving round a circle of centre O and radius r with uniform angular velocity ω and uniform speed v. At any instant t let the particle be at P where position vector is $\vec{r}$ and at $t + \Delta t$, the particle is at Q where position vector is $\vec{r}'$. Let angle $\angle POQ = \Delta\theta$ and $PQ = \vec{\Delta r}$ and the arc $PQ = \Delta l$.

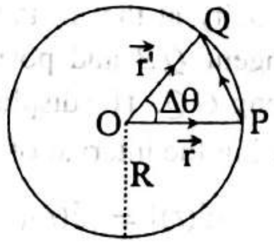


Fig. 40

Angular velocity, $\omega = (\Delta\theta/\Delta t)$; when $\Delta t \to 0$.

$\therefore\ \Delta\theta = \omega \times \Delta t$ (i)

But, $\Delta\theta = \Delta l/r$. Since Δt is very small $\Delta l = |\vec{\Delta r}| = \Delta r$

$\therefore\ \Delta\theta = \Delta r/r = \omega \times \Delta t$ [from equation (i)]

$(\Delta r/\Delta t) = r\omega$ (ii)

But, when $\Delta t \to 0$, $\dfrac{\Delta r}{\Delta t} = v$, the velocity at that instant t (iii)

From eqns. (iii) and (ii), $\boldsymbol{v = r\omega}$

Angular acceleration (α)

It is the time rate of change of angular velocity. If $\Delta\omega$ is the change in angular velocity in a time interval Δt, the angular acceleration, $\alpha = Lt_{\Delta t \to 0}(\Delta\omega/\Delta t) = (d\omega/dt)$.

Unit: rad/s^2; Dimensions: T^{-2}.

Relation between linear and angular acceleration

We know that the linear velocity is related to angular velocity as,

$$v = r\omega$$

Differentiating we have,

$$\frac{dv}{dr} = \frac{d}{dt}(\omega r) = r\frac{d\omega}{dt} \quad \therefore\ \mathbf{a = r\,\alpha}$$

Centripetal acceleration

When a particle moves round a circle with uniform speed, the direction of the velocity vector changes with time. Hence it experiences an acceleration. The direction of the acceleration is towards the centre of the circle. So it is called *central acceleration* or *centripetal acceleration.*

The *acceleration acting on an object undergoing uniform circular motion is called* **centripetal acceleration.**

Expression for centripetal acceleration

Consider a particle executing uniform circular motion along the circumference of a circle of centre O and radius r with uniform speed v and angular velocity ω.

At any instant t, let the particle be at P. Its velocity $\vec{v}(t)$ is along the tangent PA and position vector $\vec{r}(t)$ is along OP. After a small interval of time Δt, let the particle be at Q. The velocity at that instant is $\vec{v}(t+\Delta t)$ along the tangent QB and position vector is $\vec{r}(t+\Delta t)$ along OQ. The displacement $\Delta\vec{r}$ of the particle during the interval of time Δt is given by $\overrightarrow{PQ}$.

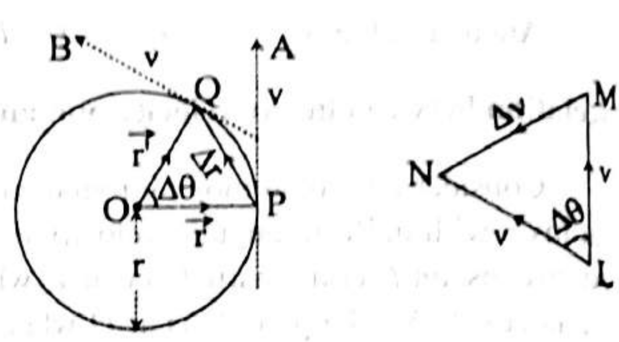


Fig. 41

$|\vec{v}(t)| = |\vec{v}(t+\Delta t)| = v,$ and

$|\vec{r}(t)| = |\vec{r}(t+\Delta t)| = r,$

If Δt is very small, $\Delta\theta$ is very small. Then the arc PQ becomes equal to the displacement. ie, $PQ = \Delta r$.

On a vector diagram, let LM and LN represent the two velocities $\vec{v}(t)$ and $\vec{v}(t+\Delta t)$ in magnitude and direction. Then $MN = \Delta\vec{v}$, represents the change in velocity in time Δt.

Triangles LMN and OPQ are similar.

$$\therefore MN/PQ = LM/OP;\ i.e., \frac{\overrightarrow{\Delta v}}{\overrightarrow{\Delta r}} = \frac{v}{r} \quad \therefore \overrightarrow{\Delta v} = \frac{v}{r} \times \overrightarrow{\Delta r} \qquad \text{(i)}$$

Dividing the equation (i) with Δt, we get,

$$(\overrightarrow{\Delta v}/\Delta t) = (v/r) \times (\overrightarrow{\Delta r}/\Delta t) \qquad \text{(ii)}$$

Limit when $\Delta t \to 0$, $(\overrightarrow{\Delta v}/\Delta t) = a$, the instantaneous acceleration at the instant t and $(\overrightarrow{\Delta r}/\Delta t) = v$, the instantaneous velocity at the instant t. Then the equation (ii) becomes,

$$\mathbf{a} = \mathbf{v}^2/\mathbf{r};\ \text{Also, } \mathbf{a} = \mathbf{v}\omega = \mathbf{r}\omega^2 (\because v = r\omega)$$

The acceleration acts in the direction of Δv *i.e.*, along MN. As Δt decreases, $\Delta\theta$ also decreases and Δv becomes smaller and smaller. Then $\Delta\vec{v}$ becomes more perpendicular to velocity vector $v(t)$ along PA and in the limiting case when Δt approaches zero, $\Delta\vec{v}$ becomes perpendicular to the velocity vector. Since the velocity vector at any point is tangential to the path at that point, the acceleration vector acts along the radius of the circle at that point and is directed towards the centre.

Examples

III.27. A stone tied to the end of a string 80 cm long is whirled in a horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 seconds, what is the acceleration? **[NCERT]**

$$r = 0.8 \text{ m}; \quad n = \frac{14}{25}; \quad a = ?$$

Central acceleration, $a = r\omega^2 = r \times (2\pi n)^2 = 4\pi^2 n^2 r$

$$= 4 \times 3.14^2 \times \left(\frac{14}{25}\right)^2 \times 0.8 = \mathbf{9.894\ ms^{-2}}$$

III.28. An aircraft executes a horizontal loop of radius 1 km with a steady speed of 900 km/h. Compare its centripetal acceleration with acceleration due to gravity. [NCERT]

$$r = 1000 \text{ m}; \quad v = 900 \text{ km/h} = 250 \text{ ms}^{-1}; \quad a = ?$$

Centripetal acceleration $a = \frac{v^2}{r} = \frac{250^2}{1000} = 62.5 \text{ ms}^{-2}$

Acceleration due to gravity, $g = 9.8 \text{ ms}^{-2}$

$$\therefore \frac{a}{g} = \frac{62.5}{9.8} = \mathbf{6.4}$$

IMPORTANT POINTS

Scalar quantity *has only magnitude and no direction*
Vector quantity *has both magnitude and direction*
A null vector is a vector with zero magnitude. It has properties

$$\vec{A} + \vec{O} = \vec{A}; \quad \lambda\vec{O} = \vec{O}; \quad \vec{O}\cdot\vec{A} = \vec{O}.$$

Addition of two vectors *by law of parallelogram or triangle of vectors. If* $\vec{A}$ *and* $\vec{B}$ *are two vectors acting at a point at an angle* θ*, the sum* $R = \sqrt{A^2 + B^2 + 2AB\sin\theta}$ *and direction is at an angle* α *with the direction of A such that* $\tan\alpha = \dfrac{B\sin\theta}{A + B\cos\theta}$.
The subtraction of a vector $\vec{B}$ *and* $\vec{A}$ *is defined as the sum of* $\vec{A}$ *and* $-\vec{B}$
Unit vector *is associated with a vector* $\vec{A}$ *and has magnitude one and is along the vector* $\vec{A}$.

$$\text{unit vector } \hat{n} = \frac{\vec{A}}{|A|}$$

$\hat{i}, \hat{j}, \hat{k}$ *are unit vectors along the x, y and z axes.*
Resolution of a vector. *A vector in a plane can be resolved in two directions x and y as* $\vec{A} = A_x\hat{i} + A_y\hat{j}$
In three dimensions $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$
Vector multiplication
Scalar product $\vec{A}\cdot\vec{B} = AB\cos\theta$ *and the product is a scalar.*
If $\vec{A}\cdot\vec{B} = 0$ *then* $\vec{A}$ *and* $\vec{B}$ *are at right angles*

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1; \quad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

Vector product. $|\vec{A}\times\vec{B}| = AB\sin\theta$. *The direction of the product is perpendicular to the plane containing* $\vec{A}$ *and* $\vec{B}$ *and given by right hand screw rule*

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Angle between two vectors $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$

Projectiles *A particle projected with a velocity u at angle* θ *with the horizontal.*

Maximum height $H = \dfrac{u^2\sin^2\theta}{2g}$ *Time of flight* $= \dfrac{2u\sin\theta}{g}$,

Range $= \dfrac{u^2\sin 2\theta}{g}$; $R_{\max} = \dfrac{u^2}{g}$

Velocity of the projectile at maximum height $= u\cos\theta$.
To obtain the same range there are two angles of projection for a given velocity.
Circular motion. *A particle moving round a circle of radius r with uniform speed v, the central acceleration* $= \dfrac{v^2}{r}$ *towards the centre.*

B. Very short answer questions

1. Distinguish between a scalar quantity and a vector quantity.
2. Pick out the only scalar quantity from the following physical quantities
   (i) Velocity (ii) Torque (iii) Area (iv) Electric current
3. Pick out the only vector quantity from the following physical quantities
   (i) Time (ii) Work (iii) Temperature (iv) Power (v) Impulse.

4. What are orthogonal unit vectors?
5. What are the angles between two vectors $\vec{A}$ and $\vec{B}$ when (i) $\vec{A} \cdot \vec{B} = 0$ and (ii) $\vec{A} \times \vec{B} = 0$
6. What do you understand by scalar product and vector product of two vectors?

C. Short answer questions

1. If $\vec{A}$ and $\vec{B}$ are two vectors, show that the magnitude of their sum $\vec{A} + \vec{B}$ is given by $|\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$, where θ is the angle between them.
2. Can two vectors of different magnitudes be combined to give a zero resultant. Can three?
3. Does it make any sense to call a quantity a vector when its magnitude is zero?
4. There is a sense of time distinguishing past, present and future. Is time a vector therefore? If not, explain why?
5. How will you obtain the sum of two vectors by triangle method?
6. i and j are unit vectors along X and Y axes respectively. What is the magnitude and direction of the vectors $i + j$ and $i - j$?

   [Ans: $\sqrt{2}$ at $\pm 45°$ with the X-axis]
7. A vector has magnitude and direction. Does it have a location in space? Can it vary with time? Will two equal vectors a and b at different locations in space necessarily have identical physical effects? Give examples in support of your answer.

   Ans. A vector has location in space. It can vary with time eg. velocity of a gas molecule is a function of time. Effects may not be the same. Eg. two equal forces acting at different points may not produce the same turning effects.
8. What is meant by horizontal range and time of flight?
9. Can there be motion in two dimensions with acceleration only in one dimension?
10. A projectile fired from the ground follows a parabolic path. Where will the speed of the projectile be minimum and why?
11. Why does the direction of motion of a projectile become horizontal at the highest point on the projectile path?
12. Explain how the vertical and horizontal components of the velocity of projection vary during the time of flight.
13. If the greatest height reached by a projectile is 1/4th the horizontal range, what is the angle of projection?
14. A person can throw a ball to a maximum height h. What is the maximum horizontal distance he can throw it?
15. Show that there are two angles of projection to obtain a certain range and these directions are equally inclined to the direction of maximum range.
16. A ball is dropped gently from the top of a tower and another ball is thrown horizontally at the same time. Which ball will hit the ground earlier? Explain why?
17. A boy sitting in a train moving at constant speed throws a ball straight up in air. Where will the ball fall?
18. What is the angular velocity of the minute-hand of a clock?

19. If a small can filled with water is rapidly swung in a vertical circle, the water does not fall down. Why?

20. A athlete can throw a ball to a maximum horizontal distance of 100 m. How high can he throw the same ball?

21. A vector has both magnitude and direction. Does this mean anything that has magnitude and direction is necessarily a vector? The rotation of a body can be specified by the direction of the axis of rotation and the angle of rotation about the axis. Does that make any rotation a vector?

22. Can three vectors not in one plane give a zero resultant? Can four vectors?

D. Essays

1. A particle is projected with a velocity u in a direction making an angle θ with the horizontal. Find (1) the maximum height (2) time of flight and (3) range on a horizontal plane through the point of projection.
2. A particle moves round a circle with constant speed. Derive an expression for the centripetal acceleration.

E. Problems

Vectors

1. Establish the following vector inequalities. [NCERT]

   (a) $|a - b| \leq |a| + |b|$
   (b) $|a - b| \geq |a| - |b|$

2. A particle has displacements 12 m towards east, 5 m towards north and 6 m vertically upwards. Find the magnitude of the sum of the displacements.

   [Ans: 14.32 m]

3. The resultant of two vectors u and v is perpendicular to u and its magnitude is half that of v. Find out the angle between u and v.

   [Ans: 150°]

4. A ship is steaming due east at a speed of 12 ms^{-1}. A passenger runs across the deck at a speed of 5 ms^{-1} towards north. What is the resultant velocity of the passenger relative to the sea?

   [Ans: 13 ms^{-1} in direction 22°37′ north of east]

5. A motor launch takes 30 s to travel 120 m upstream and 20 s to travel the same distance downstream. Calculate the speed of the current and the launch.

   [Ans: 1 ms^{-1}; 5 ms^{-1}]

6. A train is running on horizontal rails at the rate of 60 km/h and the rain drops are falling vertically with a velocity 30 km/h. Find the apparent velocity and direction of the rain relative to a passenger in the train.

   [Ans: $30\sqrt{5}$ km/h at an angle $\tan^{-1} 2$ with the vertical]

7. A boatman rows with a speed of 10 km/h in still water. If the river flows steadily at 5 km/h, in which direction should the boatman row in order to reach a point on the other bank directly opposite the point from where he started?

   [60° with the bank of the river upstream]

8. In a harbour, wind is blowing at a speed of 72 km/h and the flag on the mast of a boat anchored in the harbour flutters along N.E. direction. If the boat starts moving in the N. direction with a speed of 51 km/h, what is the apparent direction of the flag on the mast of the boat? [NCERT]

Hint: The flag is acted upon by two velocities (i) velocity 72 km/h of the wind along north-east direction (ii) velocity 51 km/h of the wind towards south produced by the motion of the ship towards north. $P = 20\ ms^{-1}$; $Q = 14.17\ ms^{-1}$; $\theta = 135°$; $\alpha = ?$

[Ans 7′ south of east]

9. A motor boat is racing towards north at 25 km/h and water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat. [NCERT]

Hint: $P = 25$ km/h; $Q = 10$ km/h; $\theta = 120°$
$R = ?$ $\alpha = ?$ [Ans: 21.8 km/h; 23°25' east of north]

10. The position of a particle is given by $\vec{r} = 3.0t\hat{i} + 20t^2\hat{j} + 5.0\hat{k}$; where t is in second and r is in metre.

Find (a) $v(t)$ and $a(t)$ of the particle and (b) the magnitude and direction of the velocity at $t = 3.0$ s. [NCERT]

Hint:

$$\vec{r} = 3.0t\hat{i} + 20t^2\hat{j} + 5.0\hat{k}$$

$$\vec{v}(t) = (dr/dt) = \mathbf{3.0i + 4.0t\hat{j}}; \quad \vec{a}(t) = (dv/dt) = \mathbf{4.0\hat{j}}$$

$$\text{At} \quad t = 3.0s;\ \vec{v} = 3.0\hat{i} + 12.0\hat{j}$$

$$|\vec{v}| = \sqrt{3^2 + 12^2} = \mathbf{12.4}\ ms^{-1};\ \tan\theta = v_y/v_x = 12/3 = 4$$

$$\theta = \mathbf{76°} \text{ with the } X\text{-axis}$$

Relative velocity

11. Two trains A and B of length 400 m each are moving on two parallel tracks in the same direction with same velocities with A ahead of B. The driver of B decides to overtake A and accelerates by $1\ ms^{-2}$. If after 50 seconds the guard of B just breaks past the driver of A what was the original distance between them? [NCERT]
Hint: $u = v_{BA} = 0$; $a = 1\ ms^{-2}$; $t = 50$ s; $S = x + 800$; $x = ?$ [Ans: 450 m]

Projectile

12. To obtain a horizontal range of 2000 m, what is the least velocity of projection?
[Ans: $140\ ms^{-1}$]

13. Find the angle of projection for which the maximum height is equal to one-fourth the horizontal range.

[Ans: 45°]

14. A shot fired horizontally from the top of a tower 176.4 m high hits the ground at a distance of 1200 m from the foot of the tower. Find the velocity of projection.
[Ans: $200\ ms^{-1}$]

15. The maximum height attained by a shell is 80 m and the horizontal range is 320 m. Find the velocity and angle of projection.

[Ans: $56\ ms^{-1}$; 45°]

16. Show that a gun will shoot three times as high when elevated at an angle of 60° as when fired at an angle of 30°, but will carry the same distance on a horizontal plane.

17. Two balls are projected from the same point at angles of projection 60° and 30° to the horizontal. If they attain the same height, what is the ratio of the velocities of projection? What is the ratio of the velocities, if they have the same horizontal range?

[Ans: $1 : \sqrt{3}$; $1 : 1$]

18. A particle rests on the top of a hemisphere of radius R. Find the smallest horizontal velocity that must be imparted to the particle, if it is to leave the hemisphere without sliding down.

[Ans: $\sqrt{Rg/2}$]

19. A boy stands at the edge of a cliff 50 m above a beach and throws a stone horizontally over the edge with a speed of 18 ms^{-1}. How long after being released does the stone strike the beach below the cliff? With what speed and angle of impact does it hit land?

[Ans: 3.19 s; 36.1 ms^{-1}; 60.1°]

20. A player kicks a football at an angle of 45° with an initial speed of 20 ms^{-1}. A second player on the goal line 60 m away in the direction of kick, starts running to receive the ball at that instant. Find the speed of the second player with which he should run to catch the ball before it hits the ground. $g = 10\ ms^{-2}$.

[Ans: $5\sqrt{2}\ ms^{-1}$]

21. A bullet fired at an angle of 30° with the horizontal hits the ground 3 km away. By adjusting its angle of projection, can one hope to hit the target 5 km away? Assume the muzzle speed to be unchanged. [NCERT]

Hint: R_{max} is only 3464 m. So the bullet will not hit the target.

Circular motion

22. Earth revolves around the sun with a speed of 30 km/s in an approximate circular path. How much acceleration does it have directed towards the sun? Orbital radius of the earth is 1.5×10^8 km.

[Ans: $6 \times 10^{-3}\ ms^{-2}$]

23. Compare the velocities of the extremities of the hour, minute and second hands of a watch, their lengths being in the ratio 6 : 10 : 3.

[Ans: 1 : 20 : 360]