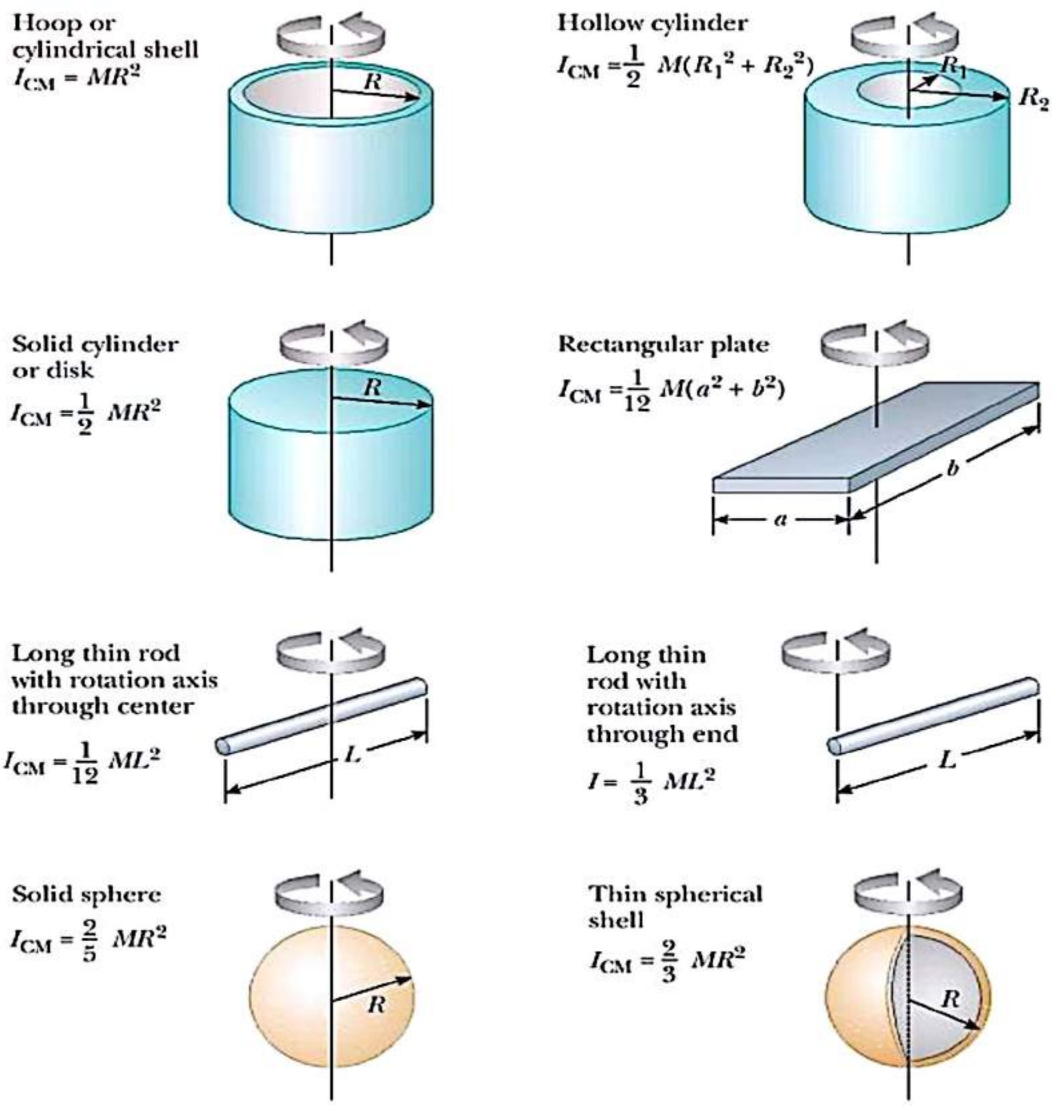
Chapter–7:

System of Particles and Rotational Motion



CBSE CLASS XI NOTES

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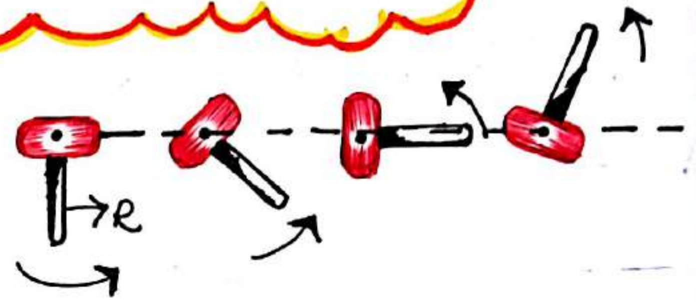
SYSTEM OF PARTICLES AND ROTATIONAL MOTION

XIth CBSE

Rigid body

It is a body with fixed geometrical size and shape, both of which do not change during motion.



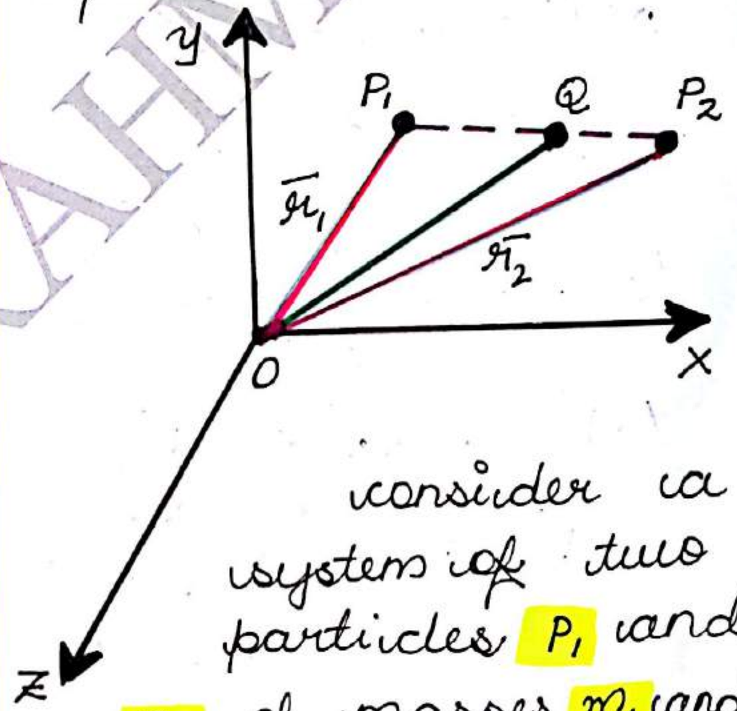
Explain centre of mass of a system of particles.

It is a point in the system where the whole mass of the system is supposed to be concentrated for describing its translatory motion.

Note:- Its motion characterises that of the system as whole. When the system moves under an external force then this point moves in the same way as a single particle would move under the same external force.

centre of mass

obtain expressions for
(i) velocity of centre of mass
(ii) acceleration of centre of mass.



consider a system of two particles P_1 and P_2 of masses m_1 and m_2 having position vectors $\bar{r}_1$ and $\bar{r}_2$ from O.

centre of mass will be at a point Q.

Its position vector

$$\bar{R} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{m_1+m_2}$$

$$\bar{R} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{m_1+m_2} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{M}$$

$$M\bar{R} = m_1\bar{r}_1 + m_2\bar{r}_2$$

Velocity of centre of mass

As two particles system move, their position vectors r_1, r_2 and R will change with respect to time

$$M\vec{R} = m_1\bar{r}_1 + m_2\bar{r}_2$$

Differentiate with respect to time

$$M\frac{d\vec{R}}{dt} = m_1\frac{d\bar{r}_1}{dt} + m_2\frac{d\bar{r}_2}{dt}$$

$$M\bar{V}_{cm} = m_1\bar{V}_1 + m_2\bar{V}_2$$

$$\bar{V}_{cm} = \frac{m_1V_1 + m_2V_2}{M}$$

Acceleration of centre of mass

Consider a system of two particles of masses m_1 and m_2 whose position vectors are $\bar{r}_1$ and $\bar{r}_2$ with respect to O.

$$\bar{R} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{M}$$

$$V_{cm} = \frac{m_1\bar{V}_1 + m_2\bar{V}_2}{M}$$

Differentiate with respect to time

$$M\bar{V}_{cm} = m_1\bar{V}_1 + m_2\bar{V}_2$$

$$M\frac{d\bar{V}_{cm}}{dt} = m_1\frac{d\bar{V}_1}{dt} + m_2\frac{d\bar{V}_2}{dt}$$

$$M\bar{a}_{cm} = m_1\bar{a}_1 + m_2\bar{a}_2$$

$$M\bar{a}_{cm} = \bar{F}_1 + \vec{F}_2 = \bar{F}_{ext}$$

$$M\bar{a}_{cm} = \bar{F}_{ext}$$

Thus centre of mass of a system moves as if it were a particle of mass equal to the total mass of system.

Prove that centre of mass of a system moves with constant velocity in the absence of external force on the system and obtain law of conservation of momentum for a centre of mass of system

$F_{ext} = M a_{cm}$

when $F_{ext} = 0$.

$\therefore M a_{cm} = 0$

$$\frac{dV_{cm}}{dt} = 0$$

$$\therefore V_{cm} = \text{constant}$$

Thus in the absence of external force, centre of mass moves with uniform velocity. According to the principle of conservation of linear momentum, when external force on a system is zero, the total linear momentum of system is conserved. It is equivalent to saying that centre of mass moves uniformly.

Discuss the motion of centre of Mass in the following systems?

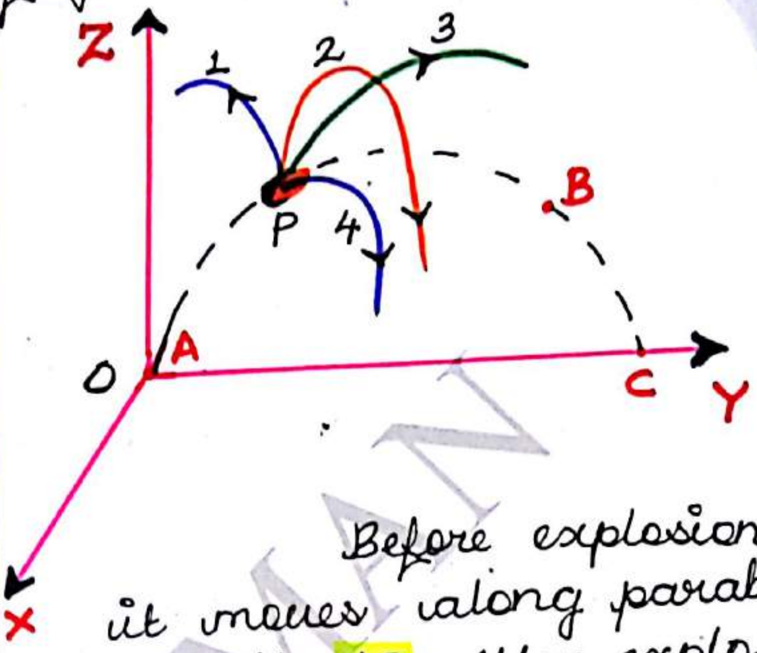
(i) Explosion of shell in flight

(ii) Spontaneous Decay of Radio active nucleus at rest

(iii) Earth-moon system

Sol; (i) Explotion of shell in flight.

Let us consider a shell which is project from the earth and explodes during flight



Before explosion it moves along parabolic path AP. After explosion, all the four fragments (shown in figure) instead of moving along parabolic path APBC, follow their own parabolic paths 1,2,3,4. The explosion is due to internal forces. After explosion, only centre of mass of four fragments will follow the parabolic path PBC.

(ii) Spontaneous Decay of Radio active nucleus at rest.

When a radio active nucleus (called parent nucleus), initially at rest, decays into two fragments, the fragments fly apart, obeying the laws of conservation of energy and momentum. However centre of mass of two fragmests remains at rest. It is because, the parent nucleus was at rest, and it decays into fragments under the effect of internal forces only.

Define Torque. Give the direction, unit and Dimension of Torque and Show that $\bar{\tau} = \bar{r}\times\bar{F}$

Torque (moment of force)

The turning effect of a force about the axis of rotation is called the moment of force or torque. and is measured as the product of magnitude of force and perpendicular distance of line of action from axis of rotation.

Its unit is Nm and dimensional formula is

ML^2T^{-2}

To show $\bar{\tau} = \vec{r}\times\vec{F}$

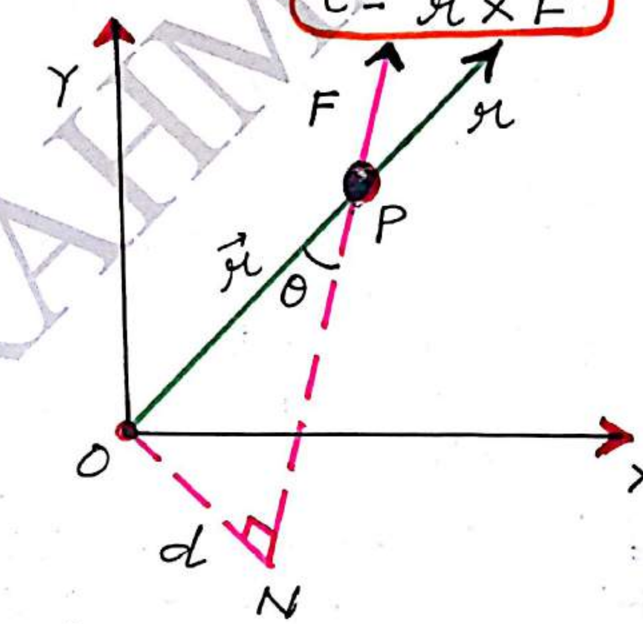
consider a particle P whose position vector is $\bar{r}$ from origin O. Let $\bar{F}$ be a force acted on P at an angle 'θ' with the direction of r.

Torque $\tau = F \times ON$

$\tau = F \times d$

$\tau = F \times r\sin\theta$

$\tau = \vec{r} \times \bar{F}$



Direction of Torque

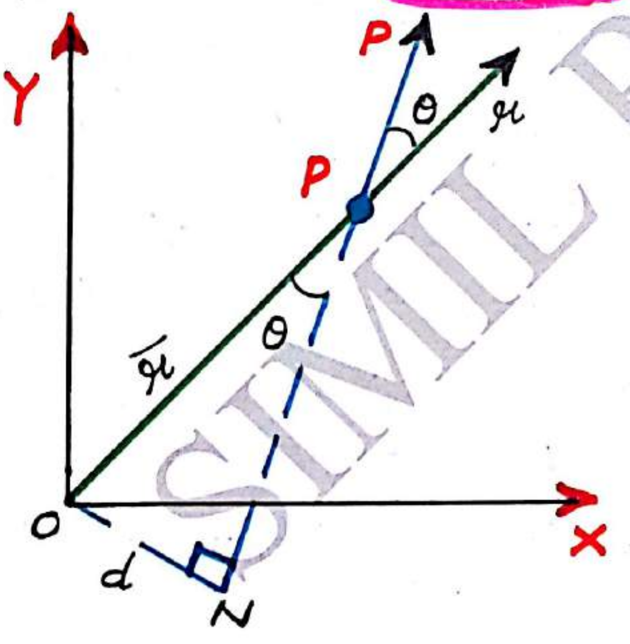
Torque direction is always perpendicular to the plane containing $\bar{r}$ and $\bar{F}$ and is given by right hand rule.

Define Angular Momentum. Give the direction, unit and dimension of Angular momentum. and show that $L = \bar{r}\times\bar{P}$

Angular momentum

The moment of linear momentum of the particle about axis of rotation is called angular momentum of the particle and is measured as the product of linear momentum and the perpendicular distance of line of action from axis of rotation. Its SI unit is $kg\,m^2 s^{-1}$. Its dimensional formula is ML^2T^{-1}

To show that $\bar{L} = \bar{r} \times \bar{P}$



consider a particle P whose position vector is $\bar{r}$ from origin O. Let $\vec{p}$ be linear momentum ($\vec{p} = m\bar{v}$) of particles which makes an angle θ with the direction of $\bar{r}$.

Angular momentum = P×ON

$$L = p \times d$$

$$L = p \times r \sin\theta$$

$$\bar{L} = \bar{r} \times \bar{P}$$

Direction of Angular momentum L

Its direction is always perpendicular to the plane containing $\bar{r}$ and $\bar{P}$ and is given by right hand rule.

What is the physical meaning of angular momentum?

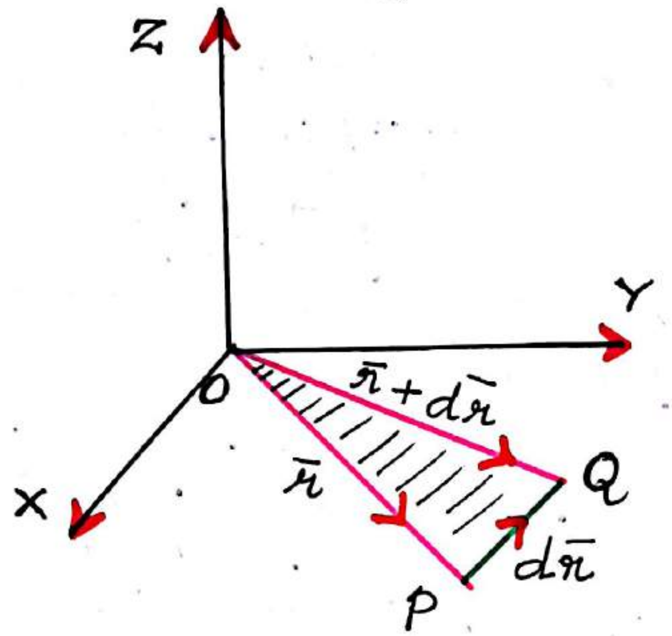
$$L = \bar{r} \times \vec{P} = rp\sin\theta$$

when $\theta = 90°$

$$L = rp$$

Angular momentum of a body about an axis is the product of linear momentum and perpendicular distance of the line of action of the linear momentum vector from axis of rotation.

What is the geometrical meaning of angular momentum (OR) prove that angular momentum is twice the product of its mass and areal velocity



consider a particle which is rotating in XY plane about Z axis. Let mass of the particle be m, and its linear momentum be $\vec{P} = m\bar{v}$

At any instant, the particle is at P so that $\overline{OP} = \bar{r}$. In a small interval of time dt, the particle reaches the position Q. $\overline{OQ} = (\bar{r} + \bar{dr})$

PQ → represents the displacement $d\bar{r}$ of the particle

$$\text{Area } OPQ = d\vec{A}$$
$$= \frac{1}{2} OP \times PQ$$
$$= \frac{1}{2}(\vec{r} \times d\vec{r})$$

dividing by dt

$$\frac{d\vec{A}}{dt} = \frac{1}{2}\bar{r} \times \frac{d\bar{r}}{dt}$$

$$\frac{d\bar{A}}{dt} = \frac{1}{2}\bar{r} \times \bar{v}$$

$$\frac{d\bar{A}}{dt} = \frac{1}{2}\bar{r} \times \frac{\bar{P}}{m} = \frac{1}{2} \times \frac{\bar{L}}{m}$$

$\bar{L}$ → angular momentum

$$\therefore L = 2m \times \frac{d\vec{A}}{dt}$$

Rotational Inertia
Moment of Inertia

The property of a body by which it resists the change in uniform rotational motion is called rotational inertia or moment of inertia.

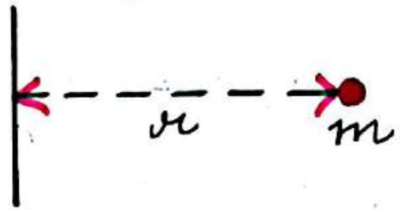
* It depends upon mass of the body.

Moment of Inertia of a particle.

Moment of Inertia of a particle about an axis is the product of the mass of the particle and the square of the distance of the particle from the axis.

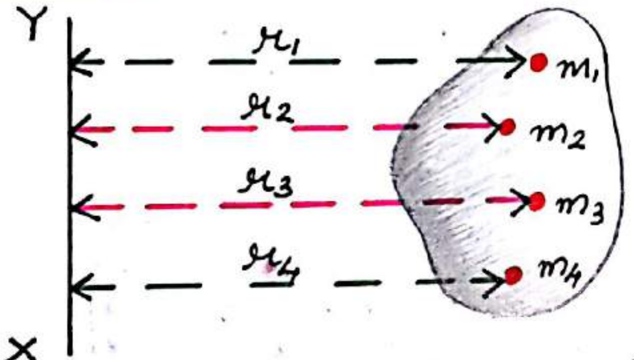
$$I = mr^2$$

m → mass
r → distance



unit : $kg\,m^2$
Dimension : ML^2

Moment of inertia of a Rigid Body.



consider a body of mass M capable of rotation about an axis XY

Let $m_1, m_2, m_3 \dots$ be the masses of various particles which are at distances $r_1, r_2, r_3 \dots$ from XY

∴ moment of Inertia

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^3 + \dots$$

i.e, $\boxed{I = \Sigma m r^2}$

Radius of gyration (K)

$\boxed{I = MK^2}$

I → moment of Inertia

K → is called Radius of gyration of the body about the axis.

It may be defined as a distance the square of which when multiplied by the mass of the body gives the moment of inertia of the body about the axis.

$$I = MK^2$$

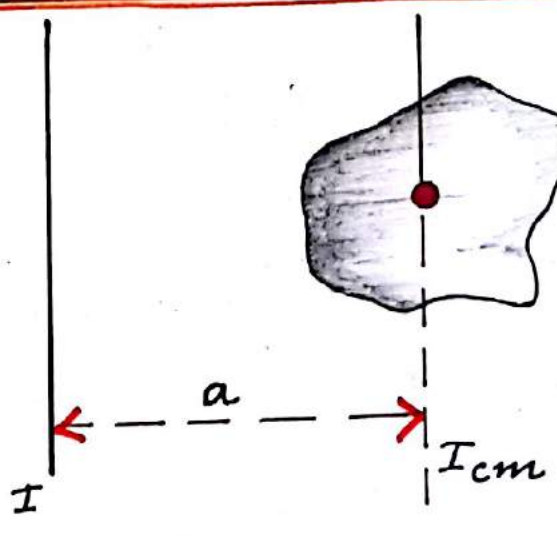
$$K^2 = \frac{I}{M}$$

$$\therefore \boxed{K = \sqrt{\frac{I}{M}}}$$

unit : m
Dimension : L

* Radius of gyration is not a constant quantity. It changes according to the location of axis of rotation.

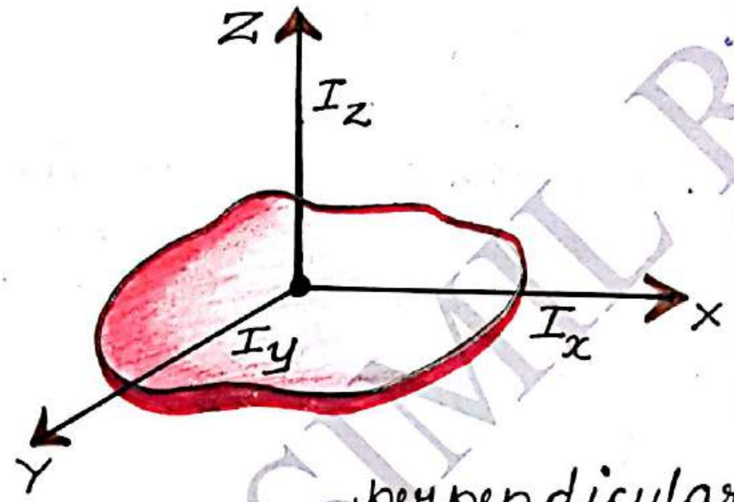
State Parallel axes theorem



The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the axes.

$$I = I_{cm} + Ma^2$$

State Perpendicular axes theorem

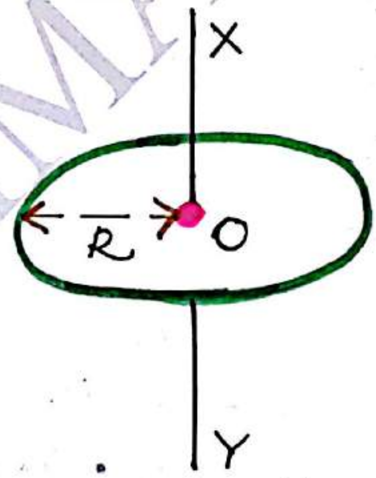


Perpendicular axes theorem states that the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moment of inertia of the lamina about any two mutually perpendicular axes in its plane and intersecting each other at the point where the perpendicular axes passes through it.

$$\therefore I_z = I_x + I_y$$

Obtain expression for moment of Inertia of a thin ring

(a) about an axis passing through centre and $\perp r$ to plane



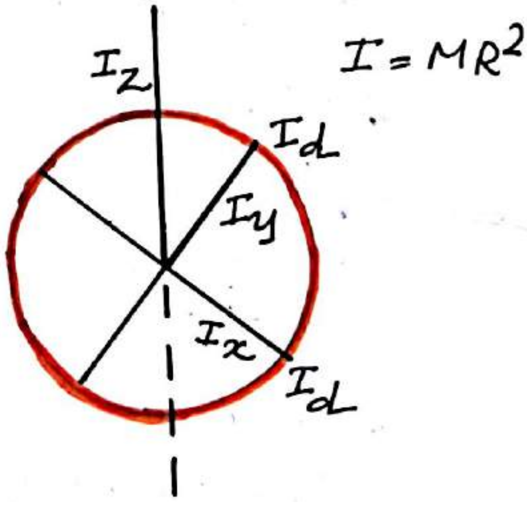
In the ring every mass is at the same distance R from axis

$$\therefore M.I = \Sigma mR^2 = MR^2$$

$$\boxed{I = MR^2}$$

(b) About diameter

$I = MR^2$



Apply perpendicular axes theorem

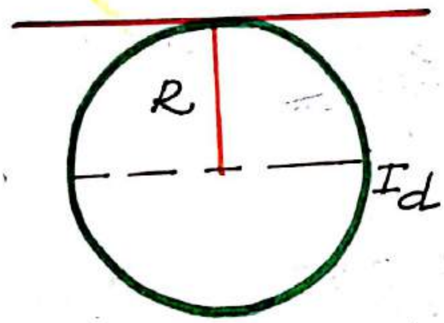
$$I_z = I_x + I_y$$

$$I = I_d + I_d$$

$$MR^2 = 2I_d$$

$$\therefore I_d = \frac{1}{2}MR^2$$

(c) About a tangent parallel to diameter



Apply parallel axis theorem

$$I = I_{cm} + Ma^2$$

$$I = I_d + MR^2$$

$$I = \frac{1}{2}MR^2 + MR^2$$

$$I = \frac{3}{2}MR^2$$

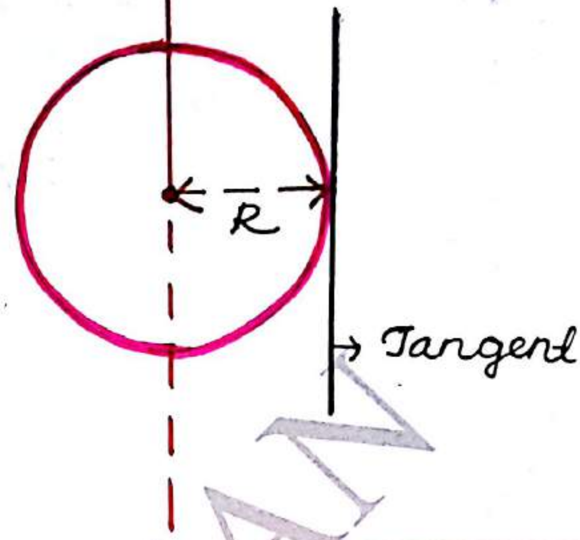
(d) About a tanget parallel to the axis passing through centre and perpendicular to the plane

$$I = I_{cm} + Ma^2$$

$$I = MR^2 + MR^2$$

$$I = 2MR^2$$

$$I = MR^2$$

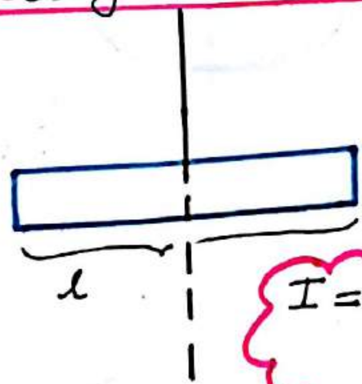


Moment of Inertia of some regular bodies - give expressions.

Rod, circular ring, circular disc, solid cylinder, hollow cylinder, solid sphere, hollow sphere.

(1) Rod.

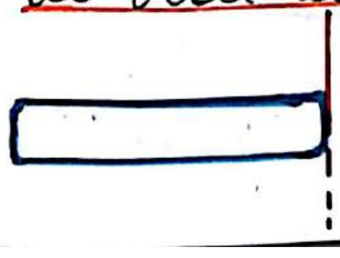
(a) about an axis $\perp r$ to rod and passing through mid point



$$I = \frac{Ml^2}{12}$$

(b) About an axis $\perp r$ to rod at one end

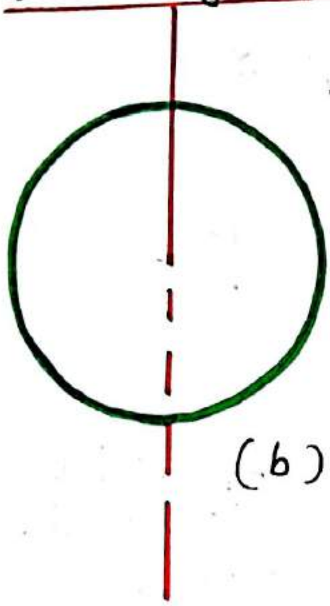
$$I = \frac{Ml^2}{3}$$



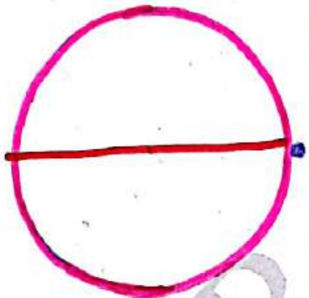
(2) Circular Ring

(a) About an axis perpendicular to plane through centre

$$I = MR^2$$



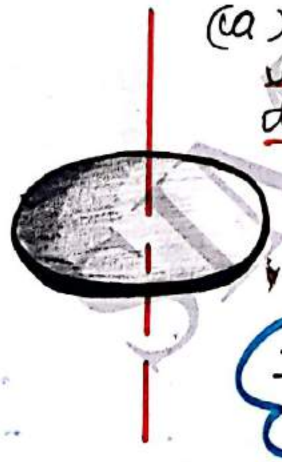
(b) About diameter



$$I = \frac{1}{2}MR^2$$

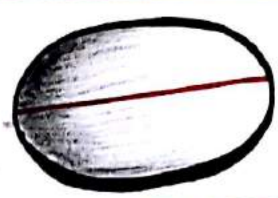
(3) Circular Disc

(a) About an axis ⊥r to plane through centre



$$I = \frac{1}{2}MR^2$$

(b) About diameter



$$I = \frac{1}{4}MR^2$$

(4) Cylinder

(a) Solid cylinder

$$I = \frac{1}{2}MR^2$$ → axis of cylinder

(b) Hollow cylinder

$$I = MR^2$$ → axis of cylinder

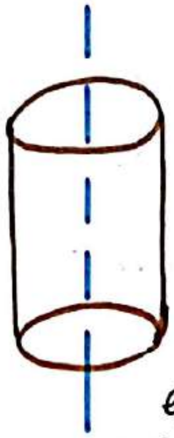
(5) Sphere

(a) Solid sphere

$$I = \frac{2}{5}MR^2$$ (Diameter)

(b) Hollow sphere

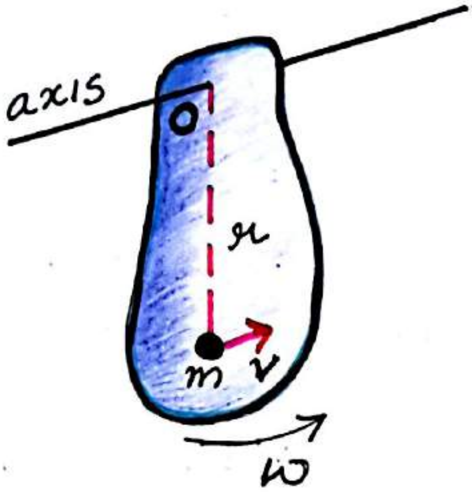
$$I = \frac{2}{3}MR^2$$ (Diameter)



extra

* Solid cylinder ⊥r to axis at the centre $I = M\left[\frac{l^2}{12} + \frac{r^2}{4}\right]$
* Hollow cylinder ⊥r to its axis at the centre $I = M\left[\frac{l^2}{12} + \frac{r^2}{2}\right]$
* Solid sphere Tangent $I = \frac{7}{5}MR^2$
* Hollow sphere Tangent $I = \frac{5}{3}MR^2$

Kinetic Energy of a rotating Body.



consider a rigid body rotating about an axis passing through O with uniform angular velocity ω. The body consists of 'n' number of particles. Let one particle of mass 'm' is situated at a distance 'r' from axis of rotation.

Linear velocity of the particle $V = r\omega$

kinetic energy of particle $= \frac{1}{2}mv^2$

$= \frac{1}{2}mr^2\omega^2$

$\therefore$ K.E of whole body $= \Sigma \frac{1}{2}mr^2\omega^2$

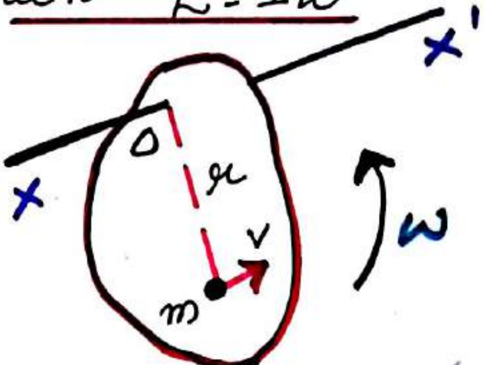
$= \frac{1}{2}\omega^2 \Sigma mr^2$

$K.E = \frac{1}{2}I\omega^2$

where $I = \Sigma mr^2$

$$\boxed{K.E_{rot} = \frac{1}{2}I\omega^2}$$

S.T angular momentum $L = I\omega$



The sum of the moments of linear momenta of all the particles of the body about the axis of rotation is called its angular momentum about that axis.

consider a rigid body rotating about an axis XOX' passing through O with uniform angular velocity ω. The body is made up of many number of particles. Let one such particle of mass m situated at a distance 'r' from axis of rotation.

Linear velocity of particle $V = r\omega$.

Linear momentum of particle $p = mV = mr\omega$

Moment of linear momentum $= r \times p$

$= mr^2\omega$

sum of moments of momenta of all particles

$= \Sigma m r^2 \omega$

$= \omega \Sigma m r^2$

$= I\omega$

$\therefore$ Angular momentum $L = I\omega$

** S.T torque is the product of moment of Inertia and angular acceleration $\tau = I\alpha$

** Also prove that torque is the rate of change of angular momentum

$\tau = \frac{dL}{dt}$

suppose that an external torque τ is applied on the rigid body about the axis OZ and as a result, an angular acceleration 'α' is produced in the rotational motion of the body. All particles have same angular acceleration, but their linear acceleration is different. The farther the particle from the axis of rotation, the greater is its linear acceleration.

Linear acceleration $a = r\alpha$

Force on the particle of mass $m = F$

$F = ma = mr\alpha$

moment of the force $= r \times f$

$= mr^2\alpha$

sum of the moment of all particles

$= \Sigma m r^2 \alpha$

$= \alpha \Sigma m r^2$

$= \alpha I$

$= I\alpha$

$\therefore$ Torque $= \tau = I\alpha$

$\tau = I\alpha$

** we know that angular momentum $L = I\omega$

differentiate w.r.t time

$$\frac{dL}{dt} = I\frac{d\omega}{dt}$$

$$= I\alpha = \tau$$

$$\therefore \tau = \frac{dL}{dt}$$

State and prove the law of conservation of angular momentum with illustration.

law of conservation of angular

momentum states that if no external torque acts on a system, the total angular momentum of the system remains conserved.

Proof:-

$$\text{Torque } \tau = \frac{dL}{dt} = 0$$

L = constant

$I\omega$ = constant

$$I_1\omega_1 = I_2\omega_2$$

Example

If an acrobat leaves a swing with his arms and legs stretched he has a certain angular velocity. If he pulls his arms and legs in, his moment of inertia becomes smaller and hence angular velocity increases.

**A rigid body rotates about an axis with uniform angular velocity ω. prove that farther the particle from the axis of rotation, the greater is its linear acceleration.

linear acceleration $a = \frac{v^2}{r}$

$$a = r\omega^2$$

here ω = constant

$$\therefore a \propto r$$

If r increases, 'a' also increases.

very short Answer Q

1. Explain the centre of mass of a system of particles?
2. Define Angular momentum?
3. state the law of conservation of angular momentum?
4. Define centripetal force
5. Define moment of inertia
6. what is meant by radius of gyration?
7. state the theorem of perpendicular axes.
8. state the theorem of parallel axes.
9. How do ballet dancers change their speed during their performance?

Long Answer type

1. Discuss the motion of centre of mass and

obtain the law of conservation of momentum

2. State and explain parallel and perpendicular axes theorems?

3. Define moment of Inertia of a body about an axis. Derive an expression for kinetic energy of a rotating body.?

4. Discuss the centre of mass systems of co-ordinates and obtain an expression for the position vector of the centre of mass.

Recap

* $R = \frac{m_1 r_1 + m_2 r_2}{M}$

$MR = m_1 r_1 + m_2 r_2$

* $V_{cm} = \frac{m_1 V_1 + m_2 V_2}{M}$

* $a_{cm} = \frac{m_1 \bar{a}_1 + m_2 \bar{a}_2}{M}$

$M a_{cm} = m_1 \bar{a}_1 + m_2 \bar{a}_2$

$M a_{cm} = F_{ext}$

* Torque $\tau = \bar{r} \times F$

* Angular momentum

$L = \bar{r} \times \bar{p}$

* Moment of Inertia

$I = mr^2$

* Radius of gyration

$I = MK^2$

$K = \sqrt{\frac{I}{M}}$

* parallel axes theorem

$I = I_{cm} + Ma^2$

* perpendicular axes theorem

$I_z = I_x + I_y$

* K.E of a rotating body

$K.E_{rot} = \frac{1}{2} I\omega^2$

* Angular momentum

$L = I\omega$

* $\tau = I\alpha$

$\tau = \frac{dL}{dt}$